Crack Identification of a Continuous Beam Like Structure in the Presence of Uncertain Modal Data by Stochastic Model Updating

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Abstract: The problem of crack identification in continuous beam like structures is considered. The cracks' locations and their depths are identified by employing experimental modal test results performed on the structure. The cracks are modeled using generic elements to include the coupling effects between shear forces and bending moments at the crack section. In the identification procedure eigen-sensitivity analysis of continuous structure is performed by implicit differentiation of structure characteristic equation. The experimentally obtained modal results are exposed to uncertainty including measurement errors, uncertainties in model order determination and etc. Uncertainty may also originate from manufacturing tolerances that are irreducible. To quantify uncertainties, a stochastic model updating is preformed on the structure using multiple sets of modal data. The crack locations and the depths are set to be unknown parameters of the model to be identified using model updating. Stochastic distributions of multiple measurements are determined and via the desired uncertainty propagation method the distribution of model modal predictions is also formed. The model random parameters are determined by matching the distributions of these two sets modal data. The identification process is mainly divided into two adjustment steps of matching the parameters mean value and their related covariance matrix. Here, the uncertainty propagation is performed by the so-called Monte-Carlo simulation for simulating the random processes.

Keywords: Crack identification, Eigen- Sensitivity, Stochastic Model Updating, Uncertainty Quantification.

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1 INTRODUCTION

Model updating is the process of correcting a mathematical model by matching the simulated and experimental obtained data. But experimentally obtained modal results are inherently exposed to uncertainty that originates from several sources including measurement errors, uncertainties in model order determination, curve fitting errors, etc. A more significant source of variability is manufacturing tolerances or disassembly and reassembly. Unlike the former, this uncertainty is irreducible. The quantification of parameter uncertainty is the main topic of stochastic model updating. It is an issue that has been neglected in the past whereas methods for uncertainty propagation have been developed extensively [1]. Propagation methods may be considered as forward procedures in comparison with quantification (or identification) methods i.e. in propagation approaches one may determine the probabilistic description of the vibration response from a posterior distribution of parameters but in a quantification problem the statistical properties of the parameters are sought from a known distribution of the response. Recent studies presented different methods for stochastic model updating. Mares et al. [2] presented a stochastic model updating technique that uses Monte-Carlo inverse propagation and multivariate multiple regression and Mottershead et al. [3] applied the method of [2] to an uncertainty quantification problem in an experimental study. The structure is a short beam manufactured from two components that connected by spot-welds. The main uncertainty of the model is concerned with spot-welds but there is also uncertainties originated from manufacturing variability, principally in the geometric tolerances.

Fonseca et al. [4] described a technique for stochastic model updating to identify and quantify variability in the parameters from experimental data by maximizing the likelihood of the measurements. They validated the method experimentally on a cantilever beam with a point mass at an uncertain location. Haddad Khodaparast et al. [5] developed a new method, based upon the perturbation procedure in two versions. The approach contains the estimation of the first and second statistical moments of randomized updating parameters from measured variability in modal responses. In their method the computing time of the stochastic model updating is reduced by making various assumptions and simplifications. In the first version of the method, the correlation between the updated parameters and measured data is omitted. This results in a procedure that requires only the first-order matrix of sensitivities. The second procedure includes this correlation (after the first iteration) but is a more expensive computation requiring the second-order sensitivities. They illustrated the stochastic model updating procedure by a physical experiment; the method is applied to the problem of determining thickness variability in a collection of plates from measured natural frequencies. Govers and Link [6] presented their approach of stochastic model updating as an extension of the classical model updating technique by an equation accounting for the statistical properties. In their approach, at first, the parameter means are updated by minimizing the difference between test and analysis output means. Next, the parameter covariance matrix is adjusted by minimizing the difference between test and analysis output covariance matrices based on their Frobenius norm. They applied the proposed method to a numerical model of a replica of the GARTEUR SM-AG19 benchmark structure [1]. The source of variability originated from disassembly and reassembly of the structure.

Haddad Khodaparast et al., defined interval model updating for the purpose of quantification of irreducible uncertainty [7]. It is a non-probabilistic method in the field of stochastic model updating that estimates the ranges of parameters instead of distributions of them. They applied the method to a frame structure with uncertain internal beams locations. Abu Husain et al. [8] employed the developed perturbation method of Haddad Khodaparast et al. [5] and discussed the subject of parameterisation for stochastic updating. The study is a further development of the method of for more complicated structures [5]. Experimental work is conducted on two sets of structures: first, simple flat plates, and second, spot welded structures. Here, a crack identification problem in continuous beam like structures is considered. Experimental modal test results are employed in order to identify crack locations and their depths. Accounting for uncertainties incorporated with the test data, it would be more relevant to perform the updating process from the stochastic point of view. Here the *covariance matrix adjustment* method of Govers and Link [6] is employed to achieve a correct description of crack locations and depths variabilities.

2 THEORY

The problem considered here is a continuous beam with multiple cracks. Fig. 1 shows such a beam with two cracks. Each crack is modeled by a generic element with a crack at half its length within the beam. So the problem of *one* continuous beam is split to multiple continuous beams that joint to each other with middling elements.

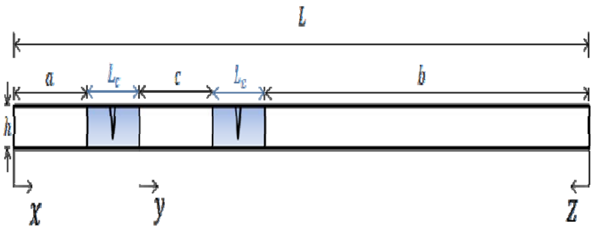


Fig. 1 A typical model of the beam with two cracks

Moreover the cracks' locations and their depths are set to be unknown parameters of the model to be identified using model updating. But the updating process is performed from the stochastic point of view. In stochastic model updating method used here, the parameters are considered as random variables and the identification process leads to identification of statistical properties of these variables i.e. mean values and standard deviations. Also sensitivity analysis of the continuous model is performed by implicit differentiation of the characteristic equation. First, the mathematical model for the beam structure is introduced and the characteristic equation is derived. Second, the stochastic model updating approach for estimation of cracks' depths and locations is discussed and then the sensitivity of the model to updating parameters is explained.

2.1. Characteristic equation

Fig. 1 shows the model of the beam with two cracks. As mentioned earlier, for each crack a generic element with a middling crack is assumed within the beam. By writing the compatibility conditions at the jointing elements the characteristic equation of the structure will be derived.
For example considering the cracked free-free beam of Fig. 1, the mode shapes of the left, middle and the right beams may be defined in general form as:

$$W_L = A_1\sinh(\lambda x)+A_2\cosh(\lambda x)+A_3\sin(\lambda x)+A_4\cos(\lambda x) \tag{1}$$

$$W_m = C_1\sinh(\lambda y)+C_2\cosh(\lambda y)+C_3\sin(\lambda y)+C_4\cos(\lambda y) \tag{2}$$

$$W_R = B_1\sinh(\lambda z)+B_2\cosh(\lambda z)+B_3\sin(\lambda z)+B_4\cos(\lambda z) \tag{3}$$

where $\lambda^4 = \frac{\rho A}{EI}\omega^2$, with '$\rho$', '$A$', '$E$' and '$I$' respectively density, cross-section area, module of elasticity and second moment of area. Also 'ω' is the unique angular frequency of the motion of the whole set at each mode i.e. the natural frequency. Applying the free boundary conditions, one obtains :

$$W_L = A_1[\sinh(\lambda x)+\sin(\lambda x)]+A_2[\cosh(\lambda x)+\cos(\lambda x)] \tag{4}$$

$$W_R = B_1[\sinh(\lambda z)+\sin(\lambda z)]+B_2[\cosh(\lambda z)+\cos(\lambda z)] \tag{5}$$

Now by focusing at jointing sections between the elements and beams showed in Fig. 2 one may write the moment and shear resultants of each section.

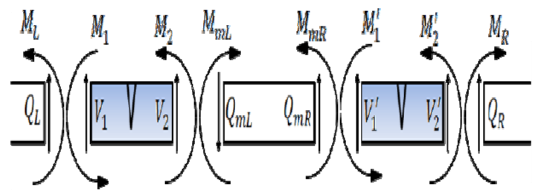


Fig. 2 Sign conventions for the moment and shear resultants of jointing sections.

$$M_L\big|_{x=a} = EI\lambda^2\{A_1[\sinh(\lambda a)-\sin(\lambda a)]+A_2[\cosh(\lambda a)-\cos(\lambda a)]\} \tag{6}$$

$$Q_L\big|_{x=a} = -EI\lambda^3\{A_1[\cosh(\lambda a)-\cos(\lambda a)]+A_2[\sinh(\lambda a)+\sin(\lambda a)]\} \tag{7}$$

$$M_{mL}\big|_{y=0} = EI\lambda^2(C_2-C_4) \tag{8}$$

$$Q_{mL}\big|_{y=0} = -EI\lambda^3(C_1-C_3) \tag{9}$$

$$M_{mR}\big|_{y=c} = EI\lambda^3[C_1\sinh(\lambda c)+C_2\cosh(\lambda c)-C_3\sin(\lambda c)-C_4\cos(\lambda c)] \tag{10}$$

$$Q_{mR}\big|_{y=c} = -EI\lambda^3[C_1\cosh(\lambda c)+C_2\sinh(\lambda c)-C_3\cos(\lambda c)+C_4\sin(\lambda c)] \tag{11}$$

$$M_R\big|_{z=b} = EI\lambda^2\{B_1[\sinh(\lambda b)-\sin(\lambda b)]+B_2[\cosh(\lambda b)-\cos(\lambda b)]\} \tag{12}$$

$$Q_R\big|_{z=b} = -EI\lambda^3\{B_1[\cosh(\lambda b)-\cos(\lambda b)]+B_2[\sinh(\lambda b)+\sin(\lambda b)]\} \tag{13}$$

Summarizing them in vector form

$$\begin{Bmatrix} Q_L \\ M_L \\ Q_{mL} \\ M_{mL} \\ Q_{mR} \\ M_{mR} \\ Q_R \\ M_R \end{Bmatrix}_{8\times1} = \begin{bmatrix} [A]_{2\times2} & & 0 \\ & [C]_{4\times4} & \\ 0 & & [B]_{2\times2} \end{bmatrix}_{8\times8} \begin{Bmatrix} A_1 \\ A_2 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ B_1 \\ B_2 \end{Bmatrix}_{8\times1} \tag{14}$$

where

$$[A]_{2\times 2}=EI\begin{bmatrix} -\lambda^3[\cosh(\lambda a)-\cos(\lambda a)] & -\lambda^3[\sinh(\lambda a)+\sin(\lambda a)] \\ \lambda^2[\sinh(\lambda a)-\sin(\lambda a)] & \lambda^2[\cosh(\lambda a)-\cos(\lambda a)] \end{bmatrix} \tag{15}$$

$$[C]_{4\times 4}=EI\begin{bmatrix} -\lambda^3 & 0 & \lambda^3 & 0 \\ 0 & \lambda^2 & 0 & -\lambda^2 \\ -\lambda^3\cosh(\lambda c) & -\lambda^3\sinh(\lambda c) & \lambda^3\cos(\lambda c) & -\lambda^3\sin(\lambda c) \\ \lambda^2\sinh(\lambda c) & \lambda^2\cosh(\lambda c) & -\lambda^2\sin(\lambda c) & -\lambda^2\cos(\lambda c) \end{bmatrix} \tag{16}$$

$$[B]_{2\times 2}=EI\begin{bmatrix} -\lambda^3[\cosh(\lambda b)-\cos(\lambda b)] & -\lambda^3[\sinh(\lambda b)+\sin(\lambda b)] \\ \lambda^2[\sinh(\lambda b)-\sin(\lambda b)] & \lambda^2[\cosh(\lambda b)-\cos(\lambda b)] \end{bmatrix} \tag{17}$$

On the other hand for the middling beam elements

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_1' \\ M_1' \\ V_2' \\ M_2' \end{Bmatrix}_{8\times 1} = \begin{bmatrix} [D_1(\lambda)]_{4\times 4} & 0 \\ 0 & [D_2(\lambda)]_{4\times 4} \end{bmatrix}_{8\times 8} \begin{Bmatrix} W_1 \\ L_c\theta_1 \\ W_2 \\ L_c\theta_2 \\ W_1' \\ L_c\theta_1' \\ W_2' \\ L_c\theta_2' \end{Bmatrix}_{8\times 1} \tag{18}$$

where $D_1(\lambda)$ and $D_2(\lambda)$ are the dynamic stiffness matrices of the generic elements defined as

$$D(\lambda)=\left[[K]-\omega^2[M]\right] \tag{19}$$

$[K]$ and $[M]$ are respectively stiffness and mass matrix of the cracked generic element and 'ω^2' can be expressed in terms of 'λ' by

$$\omega^2=\frac{EI}{\rho A}\lambda^4 \tag{20}$$

Existence of a crack in an element would not change its mass matrix $[M]$ significantly whereas would affect its stiffness matrix $[K]$ considerably. Crack depth (here shown by 'a_c') is the single parameter that affects the stiffness matrix of a cracked generic element. So dynamic stiffness matrices 'D_1' and 'D_2' respectively are functions of 'a_{c1}' and 'a_{c2}', the crack depth of each element. This issue will be discussed in detail in the section 2.2 devoted to crack modeling. Expanding the vector of nodal displacements

$$\begin{Bmatrix} W_1 \\ L_c\theta_1 \\ W_2 \\ L_c\theta_2 \\ W_1' \\ L_c\theta_1' \\ W_2' \\ L_c\theta_2' \end{Bmatrix}_{8\times 1} = \begin{bmatrix} [A']_{2\times 2} & & 0 \\ & [C']_{4\times 4} & \\ 0 & & [B']_{2\times 2} \end{bmatrix}_{8\times 8} \begin{Bmatrix} A_1 \\ A_2 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ B_1 \\ B_2 \end{Bmatrix}_{8\times 1} \tag{21}$$

where

$$[A']_{2\times 2}=\begin{bmatrix} \sinh(\lambda a)+\sin(\lambda a) & \cosh(\lambda a)+\cos(\lambda a) \\ L_c\lambda[\cosh(\lambda a)+\cos(\lambda a)] & L_c\lambda[\sinh(\lambda a)-\sin(\lambda a)] \end{bmatrix} \tag{22}$$

$$[C']_{4\times 4}=\begin{bmatrix} 0 & 1 & 0 & 1 \\ L_c\lambda & 0 & L_c\lambda & 0 \\ \sinh(\lambda c) & \cosh(\lambda c) & \sin(\lambda c) & \cos(\lambda c) \\ L_c\lambda\cosh(\lambda c) & L_c\lambda\sinh(\lambda c) & L_c\lambda\cos(\lambda c) & -L_c\lambda\sin(\lambda c) \end{bmatrix} \tag{23}$$

$$[B']_{2\times 2}=\begin{bmatrix} \sinh(\lambda b)+\sin(\lambda b) & \cosh(\lambda b)+\cos(\lambda b) \\ -L_c\lambda[\cosh(\lambda b)+\cos(\lambda b)] & -L_c\lambda[\sinh(\lambda b)-\sin(\lambda b)] \end{bmatrix} \tag{24}$$

Replacing Eq. (21) in Eq. (18) results in

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_1' \\ M_1' \\ V_2' \\ M_2' \end{Bmatrix}_{8\times 1} = \begin{bmatrix} [D_1(\lambda)]_{4\times 4} & 0 \\ 0 & [D_2(\lambda)]_{4\times 4} \end{bmatrix}_{8\times 8} \begin{bmatrix} [A']_{2\times 2} & & 0 \\ & [C']_{4\times 4} & \\ 0 & & [B']_{2\times 2} \end{bmatrix}_{8\times 8} \begin{Bmatrix} A_1 \\ A_2 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ B_1 \\ B_2 \end{Bmatrix}_{8\times 1} \tag{25}$$

Based on sign conventions shown in the Fig. 2 compatibility conditions form as

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_1^{'} \\ M_1^{'} \\ V_2^{'} \\ M_2^{'} \end{Bmatrix}_{8\times1} = \begin{Bmatrix} -Q_L \\ -M_L / L_c \\ Q_{mL} \\ M_{mL} / L_c \\ -Q_{mR} \\ -M_{mR} / L_c \\ -Q_R \\ M_R / L_c \end{Bmatrix}_{8\times1} \tag{26}$$

Finally by employing the relations between crack locations, 'x_1' and 'x_2', with respect to geometric dimensions of the structure

$$x_1 = a + \frac{L_c}{2} \tag{27}$$

$$x_2 = a + c + \frac{3L_c}{2} \tag{28}$$

The following equation forms the characteristic equation of the cracked beam

$$f\left(\lambda, x_1, a_{c1}, x_2, a_{c2}\right) = \tag{29}$$

$$Det\left(diag\left(\left[-1, \frac{-1}{L_c}, 1, \frac{1}{L_c}, -1, \frac{-1}{L_c}, -1, \frac{1}{L_c}\right]\right) \begin{bmatrix} [A]_{2\times2} & & 0 \\ & [C]_{4\times4} & \\ 0 & & [B]_{2\times2} \end{bmatrix}_{8\times8} - \begin{bmatrix} [D_1(\lambda)]_{4\times4} & 0 \\ 0 & [D_2(\lambda)]_{4\times4} \end{bmatrix}_{8\times8} \begin{bmatrix} [A^{'}]_{2\times2} & & 0 \\ & [C^{'}]_{4\times4} & \\ 0 & & [B^{'}]_{2\times2} \end{bmatrix}_{8\times8} \right) = 0$$

2.2. Crack modeling

Each crack is modeled using a generic element with a crack at half its length to include the coupling effects between shear forces and bending moments at the crack section. The characteristics of such a generic element is extracted from the paper of Ahmadian et al. [9]. The stiffness matrix of a generic beam element is generally defined as

$$K = \frac{1}{L_c^{\,3}} \begin{bmatrix} k_w & \frac{k_w}{2} & -k_w & \frac{k_w}{2} \\ & k_\theta & -\frac{k_w}{2} & \frac{k_w}{2} - k_\theta \\ & sym. & k_w & -\frac{k_w}{2} \\ & & & k_\theta \end{bmatrix} \tag{30}$$

By assigning

$$k_w = \frac{12EI}{1 + 2\left(1-\nu^2\right)\alpha^3 F_2} \tag{31}$$

$$k_\theta = \frac{EI\left[4 + \left(1-\nu^2\right)\left(18\alpha F_1 + 2\alpha^3 F_2\right)\right]}{\left[1 + 6\left(1-\nu^2\right)\alpha F_1\right]\left[1 + 2\left(1-\nu^2\right)\alpha^3 F_2\right]} \tag{32}$$

A beam element with a crack may be formed, where $\alpha = h / L_c$ is the ratio between the height and the length of the element (Here $L_c = 3h$ is assumed) and the functions F_1 and F_2 are defined as

$$F_1 = \int_0^{\frac{a_c}{h}} 2\tan\tan\left(\frac{\pi s}{2}\right)\left(\frac{0.923 + 0.199\left[1 - \sin\sin\left(\frac{\pi s}{2}\right)\right]^4}{\cos\cos\left(\frac{\pi s}{2}\right)}\right)^2 ds \tag{33}$$

$$F_2 = \int_0^{\frac{a_c}{h}} \frac{\pi s^3}{1-s}\left(3-2s\right)^2\left(1.122 - 0.561s + 0.085s^2 + 0.18s^3\right)^2 ds \tag{34}$$

where 'a_c' is the crack depth. Needless to say that 'F_1' and 'F_2' are obtained via the concepts of fracture mechanics. The crack depth 'a_c' is an unknown component in terms 'F_1' and 'F_2'. In fact 'k_w' and 'k_θ' are functions of only an unknown parameter, the crack depth 'a_c'. Thus

$$D(\lambda) = \left[[K] - \omega^2[M]\right] = \tag{35}$$

$$\frac{1}{L_c^{\,3}} \begin{bmatrix} k_w & \frac{k_w}{2} & -k_w & \frac{k_w}{2} \\ & k_\theta & -\frac{k_w}{2} & \frac{k_w}{2} - k_\theta \\ & sym. & k_w & -\frac{k_w}{2} \\ & & & k_\theta \end{bmatrix} - \left(\frac{EI}{\rho A}\lambda^4\right)\frac{\rho A L_c}{420} \begin{bmatrix} 156 & 22 & 54 & -13 \\ & 4 & 13 & -3 \\ & & 156 & -22 \\ sym. & & & 4 \end{bmatrix}$$

2.3. Stochastic model updating method

The crack locations and their depth statistical properties are estimated using stochastic model updating. Identification process is mainly divided into two adjustment steps, at first, the parameter means are updated by minimizing the difference between test and analysis output means. Next, the parameter covariance matrix is adjusted by minimizing the difference between test and analysis output covariance matrices based on their Frobenius norm. For further study see Ref. [6].

2.3.1. Parameter Mean Value Adjustment

For mean value adjustment the method of weighted least squares is used. The vector of parameter means is $\bar{p} = [\bar{x}_1 \quad \bar{a}_{c1} \quad \bar{x}_2 \quad \bar{a}_{c2}]^T$, where $\bar{x}_{1,2}$ are the crack location means (measured from one end of the beam) and $\bar{a}_{c1,2}$ are the crack depth means.

The measured mean vector consists means of the first n Eigen values $\bar{\lambda}_m = [\bar{\lambda}_{1m} \quad \bar{\lambda}_{2m} \quad \dots \quad \bar{\lambda}_{nm}]^T$ that have the following relation with the natural frequencies

$$\lambda^4 = \frac{\rho A}{EI}\omega^2 \tag{36}$$

And the corresponding Eigen values that estimated from the continuous beam model are

$$\bar{\lambda}_e = [\bar{\lambda}_{1e} \quad \bar{\lambda}_{2e} \quad \dots \quad \bar{\lambda}_{ne}]^T .$$

The parameter mean estimation starts by defining a residual which contains the differences between the estimated and measured mean Eigen values.

$$\bar{\varepsilon} = \bar{\lambda}_m - \bar{\lambda}_e \tag{37}$$

$\bar{\lambda}_e$ basically is a function of $\bar{p}$, the parameters of the model, by applying the Taylor series truncated after the linear term

$$\bar{\lambda}_e(\bar{p}_{i+1}) = \bar{\lambda}_e(\bar{p}_i) + G_i.\Delta\bar{p}_i \tag{38}$$

where $\bar{\lambda}_e(\bar{p}_i)$ or simply $\bar{\lambda}_{e_i}$ represents the vector of estimated Eigen values at the iteration step i, $G_i = \left.\frac{\partial\bar{\lambda}_e}{\partial p}\right|_i$ represents the sensitivity matrix and $\Delta\bar{p}_i$ represents the vector of the parameter changes between successive iteration steps. Thus

$$\bar{\varepsilon}_i = \bar{r}_i - G_i.\Delta\bar{p}_i \tag{39}$$

with $\bar{r}_i = \bar{\lambda}_m - \bar{\lambda}_{e_i}$. The vector $\bar{r}_i$ contains the residual between the measured and the estimated Eigen values at the i-th iteration step. The method of weighted least squares leads to the objective function for parameters mean identification

$$J_m = \bar{\varepsilon}^T W_\varepsilon \bar{\varepsilon} \to \min \tag{40}$$

where the subscript of 'J_m' denotes for mean values and 'W_ε' is a positive diagonal weighting matrix which reflects the confidence level in the frequency measurements. The estimated parameter means are obtained by minimizing 'J_m' with respect to $\Delta\bar{p}$, which involves differentiating 'J_m' with respect to each parameter, and setting the result equal to zero. The change in the parameter vector is then

$$\Delta\bar{p}_i = \left(G_i^T W_\varepsilon G_i\right)^{-1} G_i W_\varepsilon r_i \tag{41}$$

2.3.2. Parameter Covariance Matrix Adjustment

In a similar manner to mean value adjustment, the parameter covariance estimation starts by defining a residual which contains the differences between the estimated and measured covariance matrix of Eigen values.

$$\varepsilon_{Cov} = Cov_m - Cov_e \tag{42}$$

Cov_e is a function of Cov_p, the covariance of parameters, by applying the Taylor series expansion to output covariance matrix Cov_e truncated after the linear term

$$Cov_e(p_{i+1}) = Cov_e(p_i) + G_i.\Delta_{Cov_{p_i}}.G_i^T \tag{43}$$

where $Cov_e(p_i)$ or simply Cov_{e_i} represents the covariance matrix of estimated Eigen values at the iteration step 'i', 'G_i' represents the sensitivity matrix which is obtained in the same way of previous section and $\Delta_{Cov_{p_i}}$ represents the parameter covariance matrix changes between successive iteration steps. Thus

$$\varepsilon_{Cov_i} = \Delta_{Cov_i} - G_i.\Delta_{Cov_{p_i}}.G_i^T \tag{44}$$

with $\Delta_{Cov_i} = Cov_m - Cov_{e_i}$. The matrix Δ_{Cov_i} contains the residual between the measured and the estimated Eigen values covariance matrix at the i-th iteration step. The objective function for parameter covariance matrix adjustment is

$$J_c = \frac{1}{2}\left\| W_\varepsilon.\varepsilon_{Cov} W_\varepsilon^T \right\|_F^2 \to \min \tag{45}$$

with subscript 'F' denoting the Frobenius norm of a matrix, the subscript of 'J_c' denoting covariance and 'W_ε' is a positive diagonal weighting matrix which reflects the confidence level in the frequency measurements. The estimated covariance matrix of the parameters is obtained by minimizing 'J_c' with respect to Δ_{Cov_p}, which involves equation $\partial J / \partial \Delta_{Cov_p} = 0$, finally the change in the parameter covariance matrix is

$$\Delta_{Cov_{p_i}} = T_i . \Delta_{Cov_i} . T_i^T \tag{46}$$

where

$$T_i = \left(G_i . W_\varepsilon . G_i^T \right)^{-1} G_i . W_\varepsilon . \tag{47}$$

2.4. Sensitivity Analysis

Model updating process of weighted least squares method, needs the sensitivity matrix of the model. Such a matrix includes the derivatives of Eigen values with respect to model parameters, as follows

$$G = \begin{bmatrix} \frac{\partial \lambda_1}{\partial p_1} & \frac{\partial \lambda_1}{\partial p_2} & \cdots & \frac{\partial \lambda_1}{\partial p_k} \\ \frac{\partial \lambda_2}{\partial p_1} & \frac{\partial \lambda_2}{\partial p_2} & \cdots & \frac{\partial \lambda_2}{\partial p_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \lambda_n}{\partial p_1} & \frac{\partial \lambda_n}{\partial p_2} & \cdots & \frac{\partial \lambda_n}{\partial p_k} \end{bmatrix}_{n \times k} \tag{48}$$

with $n \times k$ denoting respectively number of n-first Eigen values and number of the supposed parameters of the model. This computation is performed in the following. Suppose the characteristic equation as

$$f\left(\lambda_i, p_1, p_2, \ldots, p_k \right) = 0 \tag{49}$$

where 'λ_i' is the i-th Eigen value and p's are the updating parameters. In fact 'λ_i' is the dependent variable defined explicitly by p's that are independent variables. Differentiating both sides of the characteristic equation with respect to p_j (the j-th parameter) using the chain rule results in

$$\frac{\partial f}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial p_j} + \sum_{m=1}^{k} \frac{\partial f}{\partial p_m} \frac{\partial p_m}{\partial p_j} = 0 \tag{50}$$

But p's are independent variables, so

$$\frac{\partial p_m}{\partial p_j} = \begin{cases} 1 & m = j \\ 0 & m \neq j \end{cases} \tag{51}$$

Thus

$$\frac{\partial f}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial p_j} + \frac{\partial f}{\partial p_j} = 0 \Rightarrow g_{ij} = \frac{\partial \lambda_i}{\partial p_j} = -\frac{\partial f / \partial p_j}{\partial f / \partial \lambda_i} \tag{52}$$

So the Eigen value sensitivity of the structure ends in differentiating the characteristic equation with respect to parameters and Eigen values.

3 EXPERIMENTAL STUDY

In this section an experimental example of a stochastic crack identification problem is presented. The example is selected among the experimental examples of the Ref. [10]. An aluminum free-free beam with two cracks is considered. Table 1 gives details of the geometric and material properties.

Table 1 The properties of the beam used for the experimental study

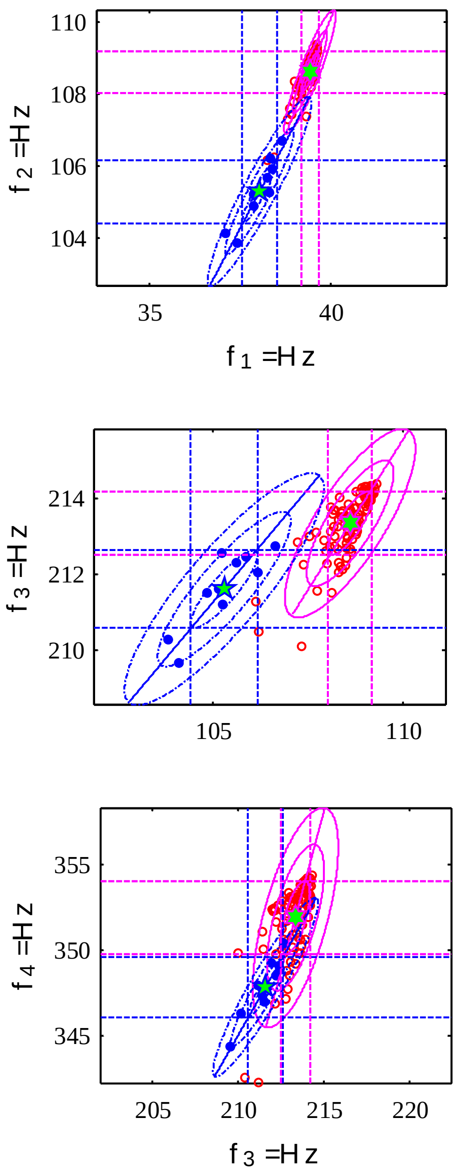
Boundary Conditions	Free-Free
Material	Aluminum
Young's modulus, E	69.79 GPa
Mass density, ρ	2600 kg/m^3
The Poisson ratio, ν	0.33
Beam length, L	1832 mm
Beam width, w	50 mm
Beam height, h	25 mm

Nominal values for mean and standard deviation of crack locations and depths are tabulated in table 2.

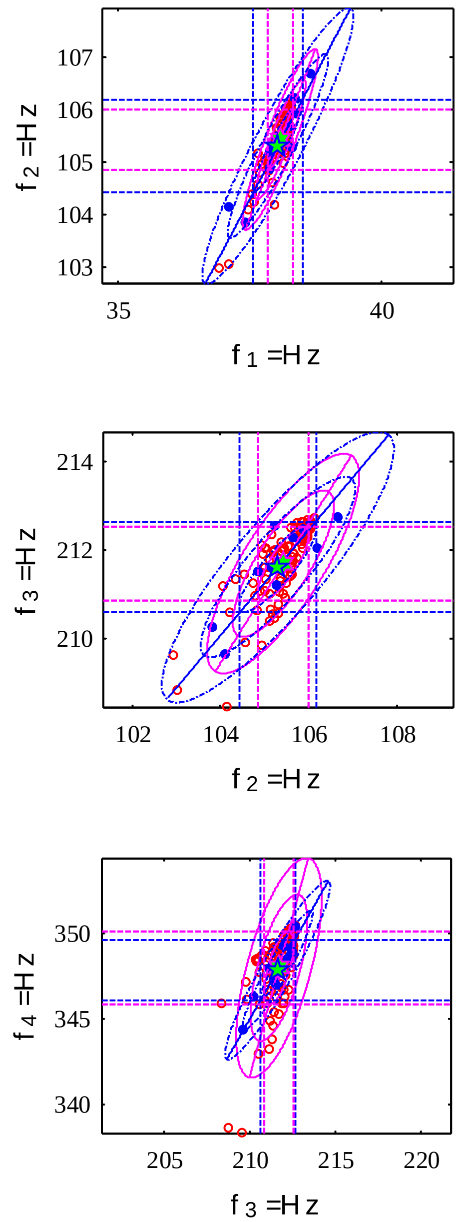
Table 2 Nominal values for statistical properties of crack locations and depths

	Location, x (mm)		Depth, a_c (mm)	
	Mean, $\bar{x}$	Std, σ_x	Mean, $\bar{a}_c$	Std, σ_{a_c}
Crack No. 1	595	4	12	1
Crack No. 2	800	5	8	1.5

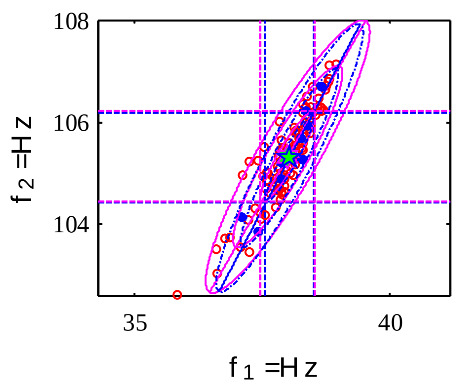
A realistic number of $n = 10$ samples are considered for the distribution of measured data whereas for the distribution of analytical model $n = 100$ samples are assumed. It is considered that the variables follow a normal distribution. Samples are generated by the so-called method of Latin Hypercube Sampling (LHS) that is much more efficient than the simple random sampling method. The uncertainty propagation is conducted by Monte-Carlo simulation to achieve the distribution of outputs of the model i.e. natural frequencies. The distribution of measurements and the initial scatter of the analytical model is shown in Fig. 3a.

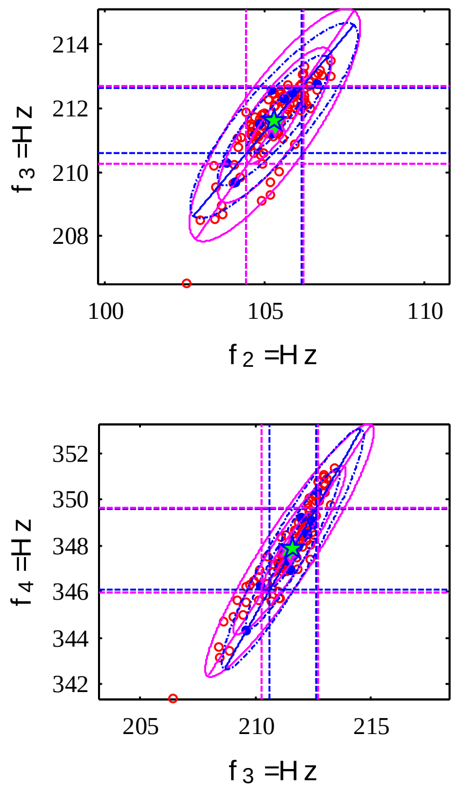


(a)



(b)





(c)

Fig. 3 Distributions and the corresponding confidence ellipses of the natural frequencies obtained by Analytical model (empty red circles) *vs.* Test data (filled blue circles): (a). Iteration step 0 (b). After mean value adjustment (c). After covariance matrix adjustment

Table 3 Statistical properties of parameters; mean values are in (*mm*) while covariances are in (mm^2)

Parameter		Nominal		Statistical	
		Test	Initial	Test ($n=10$)	Updated ($n=100$)
$\bar{p}$	$\bar{x}_1$	595	400	595.419	596.160
	$\bar{a}_{c1}$	12	4	11.959	11.924
	$\bar{x}_2$	800	700	800.300	802.388
	$\bar{a}_{c2}$	8	4	8.059	8.186
$Cov(p)$	σ^2_{x1}	4^2	8^2	19.521	25.106
	$\sigma^2_{a_{c1}}$	1^2	2^2	1.040	1.055
	σ^2_{x2}	5^2	10^2	22.045	19.683
	$\sigma^2_{a_{c2}}$	1.5^2	3^2	2.631	3.396
	$Cov(x_1, a_{c1})$	0	0	1.076	0.766
	$Cov(x_1, x_2)$	0	0	8.649	9.206
	$Cov(x_1, a_{c2})$	0	0	1.419	3.261
	$Cov(x_2, a_{c1})$	0	0	1.730	2.467
	$Cov(x_2, a_{c2})$	0	0	1.152	1.364
	$Cov(a_{c1}, a_{c2})$	0	0	3.007	6.382

By the method of Govers and Link [6] the stochastic model updating is performed and the results are shown here. Fig. 3 shows the updating process in three steps: (a) The initial scatters, (b) Distributions after parameters mean adjustment and (c) Distributions after parameter covariance matrix adjustment.

Table 3 shows all of the statistical properties for the model parameters at various stages of the updating process i.e. the mean values and covariances of the crack locations and depths for the test case, the initial analytical model and the identified one. Nominal values are those used in LHS for producing random samples.

It must be mentioned here that the distribution of the updated analytical model is generated in a correlated random space (see the Statistical column of Table 3 and note that off-diagonal elements of the covariance matrix are no longer zero). In such cases LHS is based on Cholesky decomposition [1]. Details may be found in Ref. [11].

For a better insight in how the updating process is performed, the Figs. 4 and 5 are provided which respectively correspond to the evolution of outputs mean and standard deviation.

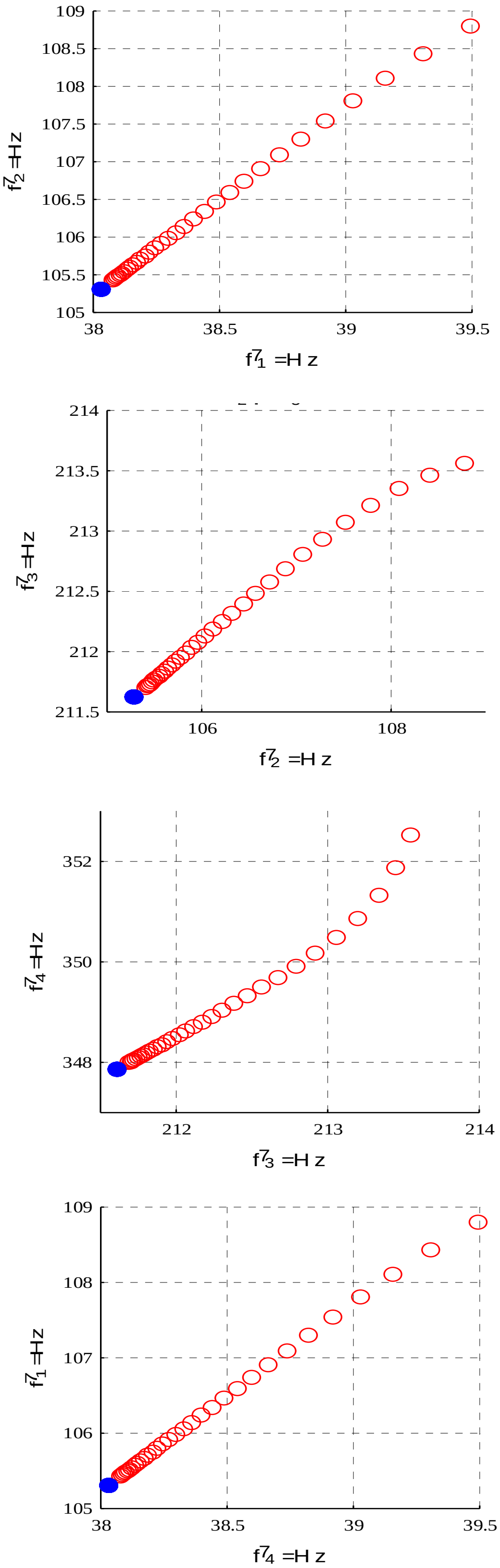


Fig. 4 Evolution of the natural frequencies mean values of the Analytical model (empty red circles) toward those of the Test data (filled blue circle)

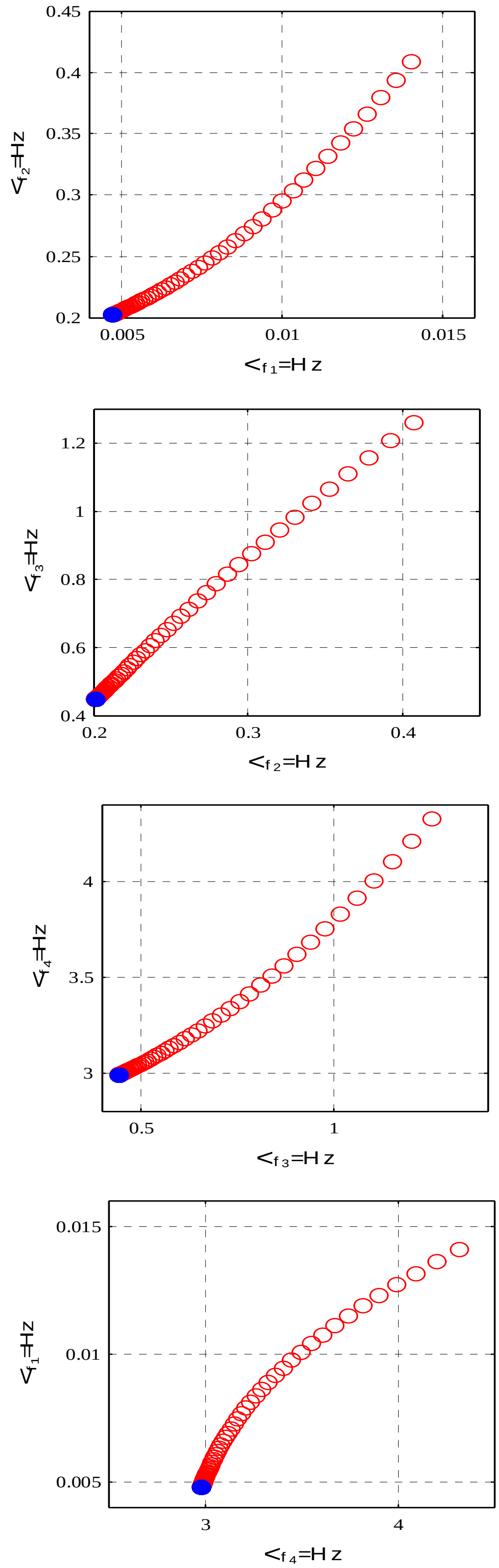


Fig. 5 Evolution of the natural frequencies standard deviations of the Analytical model (empty red circles) toward those of the Test data (filled blue circle)

Numerical results show a good agreement between test and analysis in both mean values and covariances. Statistical properties of natural frequencies of the updated analytical model concur with those of the test results as can be explicitly seen by the match of the frequency confidence ellipses. Both centers and the circumferences of the ellipses are nearly coincided to each other. Cholesky decomposition plays a crucial role in reconstructing the desired distributions.

4 CONCLUSION

In this paper the recently developed stochastic model updating technique is applied to a crack identification problem in the presence of measurement uncertainty. Statistical properties of the measured modal parameters are used in order to correct the mean values and covariances of the output of the analytical model. In this manner, the location and the depth of cracks, i.e. the model parameters, are estimated (in fact, the statistical properties of locations and depths). Random processes are conducted by the so-called Monte-Carlo simulation incorporated with Latin Hypercube Sampling (LHS) method. For the simulation of correlated random data, LHS is based on Cholesky decomposition. This is essential for achieving the correct distribution of a random space.

5 ACKNOWLEDGEMENT

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