Adaptive neural network output feedback stabilisation of nonlinear non-minimum phase systems

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ABSTRACT

This paper presents an adaptive output feedback stabilisation method based on neural networks for nonlinear non-minimum phase systems. The proposed controller comprises a linear, a neuro-adaptive, and an adaptive robustifying parts. The neural network is designed to approximate the matched uncertainties of the system. The inputs of the neural network are the tapped delays of the system input-output signals. In addition, an appropriate reference signal is proposed to compensate the unmatched uncertainties inherent in the internal system dynamics. The adaptation laws for the neural network weights and adaptive gains are obtained using the Lyapunov's direct method. These adaptation laws employ a linear observer of system dynamics that is realisable. The ultimate boundedness of the error signals are analytically shown using Lyapunov's method. The effectiveness of the proposed scheme is shown by applying to a translation oscillator rotational actuator (TORA) model**.**

Keywords: Adaptive control, Neural networks, Nonlinear systems, Output feedback, Robust control, Stabilisation

1 Introduction

Output feedback control for nonlinear systems is a challenging problem in control theory that has been studied in recent years. Various methods have been proposed in this area including using geometric techniques [1], adaptive observers and output feedback controllers for system in output feedback form [2], high gain observers [3], backstepping method for nonlinear systems with parametric uncertainties [4], and combining the backstepping with the small gain theorem [5]. The main objective of these efforts is to propose systematic design methods for controlling nonlinear systems in the presence of structured uncertainties in the form of parameters variations and unstructured uncertainties such as unmodeled dynamics.

Many works have been presented based on the output feedback control method using neural networks (NNs) techniques. These methods can be applied to a wide class of nonlinear systems with structured and unstructured uncertainties. Some methods have been focused on control of uncertain systems using high-gain observers [6, 7], adaptive error observers [8], and constructing a strictly positive real (SPR) error dynamics and using Kalman-Yakobuvich lemma [9, 10]. These methods are based on the minimum phase assumption. The minimum phase assumption guarantees global asymptotic stability of the zero dynamics. In fact, for controlling a non-minimum phase system, information on the state variable associated with the zero dynamics is normally required. State observation of nonlinear systems is often not straightforward task, particularly for a complex nonlinear system. However, many methods of nonlinear systems yield a linear observer error dynamics [11]. A particular case for the systems in output feedback form can be found in [2].

Recently, many papers have dealt with output feedback stabilisation of non-minimum phase systems. Isidori [12] has presented a solution for robust semi-global output feedback stabilisation of non-minimum phase systems based on auxiliary constructions using a high-gain observer. Karagiannis *et al.* [13] have proposed a method for global output feedback stabilisation by designing an observer and using the classical backstepping and the small-gain techniques. A design method for semi-global stabilisation of a class of non-minimum phase nonlinear systems, which can be transformed to the global normal form and to the form of linear observer error dynamics, has been presented by Ding [11]. These methods have considered the stabilisation problem for nonlinear systems in which their nonlinearities and the high frequency gain depend only on the output of the system. Various results on local and non-local stabilisation of non-minimum phase nonlinear systems have been presented that deal with more general nonlinear systems using the universal approximation property of neural networks and fuzzy systems [14, 15]. However, in these works, it has been assumed that the system states are available. Hovakimyan *et al.* [16] have considered the output feedback control of a more general class of non-minimum phase nonlinear system based on universal approximation properties of NN and using a high order error observer.

This paper presents an adaptive output feedback stabilisation method for observable and stabilisable nonlinear non-minimum phase systems, where the matched uncertainties are cancelled out using NNs. In contrary to analytical approaches presented in [11-13], in the proposed method, the internal dynamic and the high frequency gain of system is not restricted to be dependent only on the system output. Moreover, it is an observer-based method; hence, only the system output is required for designing the controller. In addition, the stability result is semi-global in the sense that it is local with respect to the compensation domain of the matched and unmatched uncertainties.

For designing the controller, a linear approximation model of the nonlinear system is first derived to represent the non-minimum phase zeros of the nonlinear system with desired accuracy. In fact, there is a conic sector bound on the modelling error of the non-minimum phase zeros which is referred to as the unmatched uncertainty. Hence, the proposed approach can be applied to uncertain systems, which have partially known Lipschitz continuous functions in their arguments. Then, a static linear controller is proposed to stabilise the linear part of the system in the absence of nonlinearities. Finally, a controller, which is obtained from this linear controller and a neuro-adaptive element, is used to approximate the matched uncertainty. The NN operates over a tapped-delay line of memory units, comprised of the system input/output signals [16]. In addition, in comparison with [16], an extra part, which is termed as a robustifying control part, is included in the control law to compensate the NN approximation error. Moreover, to achieve a semi-global stabilisation, the unmatched uncertainties, inherent in the internal dynamics of the system, is cancelled out using a suitable reference signal. For realisation of the proposed control method, a linear observer is proposed to estimate the error dynamics of the system. Hence, it is assumed that only the system output is measurable.

This paper is organised as follows: Section 2 describes the class of nonlinear systems which is considered in this paper. In this section the problem of stabilisation associated with these systems is also clarified. The procedure of the control and observer design, and approximation properties of the NN are addressed in Section 3. Section 4 provides analytical results on stability of the closed-loop system. Simulation results are presented in Section 5. Conclusions are given in Section 6.

In this paper, $\lambda_{\min}(\mathbf{P})$ and $\lambda_{\max}(\mathbf{P})$ stand for the minimum and maximum eigenvalues of the symmetric matrix P, respectively.

2 Problem statement

Consider the nonlinear system [1]

$$\begin{cases} \dot{z}_i = z_{i+1} & 1 \le i \le r-1 \\ \dot{z}_r = f(\mathbf{z}, \boldsymbol{\eta}, u) & \\ \dot{\eta}_j = \dot{\eta}_{j+1} & 1 \le j \le n-r-1 \\ \dot{\eta}_{n-r} = v(\mathbf{z}, \boldsymbol{\eta}) & \\ y = z_1, & \end{cases} \tag{1}$$

which is in the normal form with the coordinates $\left[\mathbf{z}^T, \boldsymbol{\eta}^T\right] = \left[z_1, \ldots, z_r, \eta_{r+1}, \ldots, \eta_n\right]$ where r $(1 \le r < n)$ is the relative degree, $\boldsymbol{\eta} \in \Omega_\eta \subset R^{n-r}$ is the state vector associated with the internal dynamics, $\mathbf{z} \in \Omega_z \subset R^r$ where Ω_η and Ω_z are the compact sets of operating regions, and u and y are the input and the output of the system, respectively. The mappings $f: R^{n+1} \to R$ and $v: R^n \to R$ are partially known and continuous Lipschitz functions with initial conditions $f(\mathbf{0}, \mathbf{0}, 0) = 0$ and $v(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. Note that the system (1) can be non-minimum phase. Hence, the stability assumption on the zero dynamics of the system is not required.

Assumption 1. *Assume that* $f_u = \partial f(\mathbf{z}, \boldsymbol{\eta}, u)/\partial u \neq 0 \quad \forall u \in R$. *This condition implies that the smooth function* f_u *is strictly either positive or negative on the compact set*

$$U = \left\{(\mathbf{z}, \boldsymbol{\eta}, u) \middle| \mathbf{z} \in \Omega_z, \ \boldsymbol{\eta} \in \Omega_\eta, \ u \in R\right\}.$$

It is also assumed that only the system output $y(t)$ is measurable.

The objective is to design an appropriate adaptive control for stabilising of the entire system (1) including the internal dynamics.

3 Controller design

3.1 Construction of error dynamics

Using the Taylor expansion, the system in (1) can be represented as

$$\begin{cases} \dot{z}_i = z_{i+1} & 1 \le i \le r-1 \\ \dot{z}_r = \mathbf{m}^T \mathbf{z} + \mathbf{n}^T \boldsymbol{\eta} + bu + b\Delta(\mathbf{z}, \boldsymbol{\eta}, u) \\ \dot{\eta}_j = \dot{\eta}_{j+1} & 1 \le j \le n-r-1 \\ \dot{\eta}_{n-r} = \mathbf{f}^T \boldsymbol{\eta} + \mathbf{g}^T \mathbf{z} + \Delta_{\boldsymbol{\eta}}(\mathbf{z}, \boldsymbol{\eta}) \\ y = z_1, \end{cases} \tag{2}$$

where **m**, **n**, **f** and **g** are coefficient vectors with appropriate dimensions, and $\Delta(\mathbf{z}, \boldsymbol{\eta}, u)$ and $\Delta_{\boldsymbol{\eta}}(\mathbf{z}, \boldsymbol{\eta})$ are unknown functions, which are referred to as the matched and unmatched uncertainties, respectively [17].

Define the error signal $e = e_1 \triangleq y_d - y$ where y_d is the reference signal and consider the combined control law as

$$u = u_L - u_{ad} - u_R + \frac{1}{b} y_d^{(r)} - \frac{\mathbf{m}^T}{b} \mathbf{y}_d \tag{3}$$

where $\mathbf{y}_d = \begin{bmatrix} y_d & \cdots & y_d^{(r-1)} \end{bmatrix}^T$ and u_L, u_{ad} and u_R are the linear, the adaptive and the robustifying control parts, respectively. Then, the error dynamics can be described as

$$\begin{cases} \dot{e}_i = e_{i+1} & 1 \le i \le r-1 \\ \dot{e}_r = \mathbf{m}^T \mathbf{e} - \mathbf{n}^T \boldsymbol{\eta} - bu_L + b(u_{ad} - \Delta + u_R) \\ \dot{\eta}_j = \dot{\eta}_{j+1} & 1 \le j \le n-r-1 \\ \dot{\eta}_{n-r} = \mathbf{f}^T \boldsymbol{\eta} - \mathbf{g}^T \mathbf{e} + y^* + \Delta_{\boldsymbol{\eta}}(\mathbf{z}, \boldsymbol{\eta}) \end{cases} \tag{4}$$

where $y^* = g_1 y_d + g_2 \dot{y}_d + \cdots + g_r y_d^{(r-1)}$, $\mathbf{g} = \begin{bmatrix} g_1 & \cdots & g_r \end{bmatrix}^T$, $\mathbf{e} = \begin{bmatrix} e_1 & \cdots & e_r \end{bmatrix}^T$ and $e_i = y_d^{(i-1)} - z_i \; (2 \le r \le r)$.

Assumption 2. *The signal* y_d *and its derivatives, and the unmatched uncertainty* $\Delta_{\boldsymbol{\eta}}(\mathbf{z}, \boldsymbol{\eta})$ *are bounded with constant and a conic sector bounds, respectively. That is*

$$\left.\begin{aligned} &\left|\Delta_{\boldsymbol{\eta}}(\mathbf{z}, \boldsymbol{\eta})\right| \le c_0^* + c_1 \|\mathbf{z}\| + c_2^* \|\boldsymbol{\eta}\| \\ &\sum_{i=0}^{r} \left| y_d^{(i)} \right| \le c_3^* \end{aligned}\right\} \tag{5}$$

where c_0^*, c_2^* *and* c_3^* *are unknown constants and* c_1 *is a known positive constant such that* $c_1 < 1$.

Remark 1: In many stabilisation approaches of non-minimum phase systems, it is assumed that the internal dynamics only depend on the system output y [11]-[13]. In this paper, a more general case is proposed. In this method, the internal dynamics of the system are not restricted only to the system output. In a particular case, when $g_i = 0\,(i = 2, \ldots, r)$ and $\Delta_{\boldsymbol{\eta}}$ is a function of z_1 instead of $\mathbf{z}$, the internal dynamics depend only on the system output.

Let $\boldsymbol{\xi} \triangleq [\mathbf{e}^T, \boldsymbol{\eta}^T]^T$. The error dynamics of the nonlinear system (4), can be represented as

$$\begin{cases} \dot{\boldsymbol{\xi}} = \mathbf{A}\boldsymbol{\xi} + \mathbf{b}\, u_L + \mathbf{b}\left(\Delta - u_{ad} - u_R\right) + \mathbf{q}\left(y^* + \Delta_{\eta}\right) \\ e = \mathbf{c}\,\boldsymbol{\xi} \end{cases} \tag{6}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{M} & -\mathbf{N} \\ -\mathbf{G} & \mathbf{F} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} \mathbf{0}_{1\times(r-1)} & -b & \mathbf{0}_{1\times(n-r)} \end{bmatrix}^T, \mathbf{q} = \begin{bmatrix} \mathbf{0}_{n-1} & 1 \end{bmatrix}^T, \mathbf{c} = \begin{bmatrix} 1 & \mathbf{0}_{n-1} \end{bmatrix},$$

$$\mathbf{M} = \left[\begin{array}{cc} \mathbf{0}_{(r-1)\times 1} & \mathbf{I}_{(r-1)\times(r-1)} \\ \hline \multicolumn{2}{c}{\mathbf{m}^T} \end{array}\right], \ \mathbf{N} = \begin{bmatrix} \mathbf{0}_{(r-1)\times(n-r)} \\ \mathbf{n}^T \end{bmatrix}, \mathbf{F} = \left[\begin{array}{cc} \mathbf{0}_{(n-r-1)\times 1} & \mathbf{I}_{(n-r-1)\times(n-r-1)} \\ \hline \multicolumn{2}{c}{\mathbf{f}^T} \end{array}\right], \ \mathbf{G} = \begin{bmatrix} \mathbf{0}_{(n-r-1)\times r} \\ \mathbf{g}^T \end{bmatrix}$$

Since the system is non-minimum phase, $\mathbf{A}$ has at least one eigenvalue with the positive real part. Therefore, the linear control u_L is first designed to stabilise the associated linearised system.

3.2 Linear control design

Controllability of $(\mathbf{A}, \mathbf{b})$ ensures the existence of the unique symmetric positive-definite solution $\mathbf{P}_1$ of the algebraic Riccati equation

$$\mathbf{P}_1\mathbf{A} + \mathbf{A}^T\mathbf{P}_1 + \mathbf{Q}_1 - 2\mathbf{P}_1\mathbf{b}\mathbf{b}^T\mathbf{P}_1 = 0 \tag{7}$$

where $\mathbf{Q}_1$ is an arbitrary symmetric positive-definite matrix and $r > 0$ is a given number. The (optimal) linear control is

$$u_L = -\rho_1 = -\mathbf{k}_c\,\hat{\boldsymbol{\xi}} \tag{8}$$

where $\hat{\boldsymbol{\xi}}$ denotes estimation of $\boldsymbol{\xi}$ and the vector gain $\mathbf{k}_c$ is

$$\mathbf{k}_c^T = \mathbf{P}_1\mathbf{b} \tag{9}$$

Substituting (9) into (7) gives

$$\left(\mathbf{A} - \mathbf{b}\mathbf{k}_c\right)^T \mathbf{P}_1 + \mathbf{P}_1\left(\mathbf{A} - \mathbf{b}\mathbf{k}_c\right) + \mathbf{Q}_1 = 0 \tag{10}$$

Hence, $\mathbf{A} - \mathbf{b}\mathbf{k}_c$ is a stable matrix and u_L stabilises the system (6) if $u_{ad} + u_R$ and y^* cancel out the matched and unmatched uncertainties, respectively.

3.3 Neural network-based adaptive controller design

The function $\Delta(\mathbf{z}, \boldsymbol{\eta}, u)$ is unknown and it must be approximated for designing an appropriate stabilising control. In fact, $\Delta(\mathbf{z}, \boldsymbol{\eta}, u)$ is approximated by employing an appropriate multilayer perceptron (MLP) to construct a suitable adaptive part u_{ad} of the control law in (3) for eliminating the influence of the unknown signal $\Delta(\mathbf{z}, \boldsymbol{\eta}, u)$ on the system. Hence, there exists a fixed-point problem as

$$u_{ad}(t) = \Delta(\mathbf{z}, \boldsymbol{\eta}, -u_{ad} + u_{\alpha}) \tag{11}$$

where $u_{\alpha} = u_L - u_R + \frac{1}{b} y_d^{(r)} - \frac{\mathbf{m}^T}{b} \mathbf{y}_d$.

According to the contractive mapping theorem [18], if the map $u_{ad} \to \Delta$ is contractive over the entire input domain then the fixed point problem (20) has a unique solution for u_{ad}. On the other hand, this map is contractive if it ensures the following condition

$$\left| \frac{\partial \Delta}{\partial u_{ad}} \right| < 1 \tag{12}$$

Substituting (1), (2) and (3) into (12), yields

$$\left| \frac{\partial \Delta}{\partial u_{ad}} \right| = \left| \frac{1}{b} \frac{\partial \left(f(\mathbf{z}, \boldsymbol{\eta}, u) - \mathbf{m}^T \mathbf{z} - \mathbf{n}^T \boldsymbol{\eta} - bu \right)}{\partial u} \frac{\partial u}{\partial u_{ad}} \right| = \left| 1 - \frac{1}{b} \frac{\partial f}{\partial u} \right| < 1 \tag{13}$$

The condition (13) is equivalent to the following two simultaneous conditions

$$\operatorname{sgn}(b) = \operatorname{sgn}(\partial f / \partial u), \quad |b| > 0.5 |\partial f / \partial u| \tag{14}$$

Under the observability condition of the system (1), it can be shown that the continuous-time dynamic $\Delta(\mathbf{z}, \boldsymbol{\eta}, u)$ can be approximated using the delayed inputs and outputs as [10, 19]

$$\Delta(\mathbf{z}, \boldsymbol{\eta}, u) = \Gamma(\boldsymbol{\zeta}) + \varepsilon_0 \tag{15}$$

where $\boldsymbol{\zeta} = [1 \quad \bar{\mathbf{y}} \quad \bar{\mathbf{u}}]^T \in R^N$ and $\Gamma : R^N \to R$ is a smooth continuous function and

$$\bar{\mathbf{u}} = [u(t - T_d) \quad \cdots \quad u(t - T_d(n_1 - r - 1))], \; n_1 \geq n$$
$$\bar{\mathbf{y}} = [y(t) \quad \cdots \quad y(t - T_d(n_1 - 1))]$$

and Γ is an update function which is obtained using the NN techniques. The approximation error ε_0 is directly proportional to the sampling time interval T_d. Hence, ε_0 can be ignored by selecting a sufficiently small $T_d > 0$. Note that the sampling time T_d is always a positive real number even if it is selected sufficiently small.

On the other hand, standard MLPs (particularly with functional inputs) are universal approximators and they can approximate any sufficiently smooth function on an appropriate compact set with any desired degree of accuracy and an arbitrarily bounded error [20]-[22]. Moreover, the domain of the function $\Delta(\mathbf{z},\boldsymbol{\eta},u)$ is the compact set R with the Euclidean norm and also any closed and bounded subset of R is also a compact set. Therefore, based on these facts, a set of ideal weights $\mathbf{w}^*$ and $\mathbf{V}^*$ on the compact set Ω_ζ exists such that

$$\Delta\left(\mathbf{z},\boldsymbol{\eta},u\right) = \mathbf{w}^{*T}\boldsymbol{\sigma}(\mathbf{V}^{*T}\boldsymbol{\zeta}) + \varepsilon_1 \qquad \forall\boldsymbol{\zeta}\in\Omega_\zeta \tag{16}$$

where $\mathbf{V}^* \in R^{N\times m}$ and $\mathbf{w}^* \in R^m$ are synaptic weights connecting the input layer to the hidden layer and the hidden layer to the output layer, respectively $\boldsymbol{\sigma} = \left[\sigma_1 \cdots \sigma_m\right]^T$ is a vector function containing nonlinear functions of neurons in the hidden layer, and $|\varepsilon_1| \le \varepsilon_{1M}$ in which ε_{1M} depends on the network architecture. The ideal constant weights $\mathbf{w}^*$ and $\mathbf{V}^*$ are defined as

$$\left(\mathbf{w}^*,\mathbf{V}^*\right) \triangleq \arg \min_{(\mathbf{w},\mathbf{V})\in\Omega_{\mathrm{w}}} \left\{ \sup_{\boldsymbol{\zeta}\in\Omega_\zeta} \left| \mathbf{w}^T\boldsymbol{\sigma}\left(\mathbf{V}^T\boldsymbol{\zeta}\right) - \Gamma(\boldsymbol{\zeta}) \right| \right\} \tag{17}$$

where $\Omega_{\mathrm{w}} = \left\{ (\mathbf{w},\mathbf{V}) \middle| \|\mathbf{w}\| \le M_{\mathrm{w}}, \|\mathbf{V}\|_{\mathrm{F}} \le M_{\mathrm{V}} \right\}$, in which M_{w} and M_{V} are positive numbers, and $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm. Since Δ can be modelled using a NN, an MLP is employed to construct the adaptive control part as

$$u_{ad} = \mathbf{w}^T\boldsymbol{\sigma}\left(\mathbf{V}^T\boldsymbol{\zeta}\right) \tag{18}$$

However, in practice, the weights of this NN may be different from the ideal ones, defined in (17). Therefore, an approximation error is required to remove this obstacle.

Lemma 1: *Let the activation functions σ_i $(i = 1,\ldots, m)$ be logistic or hyperbolic tangent functions. Then, the approximation error, which arises from the difference between (16) and (18), satisfies the following equality*

$$\Delta(\mathbf{z},\boldsymbol{\eta},u) - u_{ad} = \tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma} - \dot{\boldsymbol{\sigma}}\mathbf{V}^T\boldsymbol{\zeta}\right) + \mathrm{tr}\left(\tilde{\mathbf{V}}^T\boldsymbol{\zeta}\mathbf{w}^T\dot{\boldsymbol{\sigma}}\right) + \delta(t) \tag{19}$$

where $\tilde{\mathbf{w}} = \mathbf{w}^* - \mathbf{w}$, $\tilde{\mathbf{V}} = \mathbf{V}^* - \mathbf{V}$, $|\delta(t)| \le \varepsilon_{1M} + 2\sqrt{m}M_{\mathrm{w}} + \alpha M_{\mathrm{w}} \|\tilde{\mathbf{V}}\|_{\mathrm{F}} \|\boldsymbol{\zeta}\| + \alpha M_{\mathrm{V}} \|\tilde{\mathbf{w}}\|\|\boldsymbol{\zeta}\|$ *and*

$$\dot{\boldsymbol{\sigma}} = \begin{bmatrix} \dot{\sigma}_1(v_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dot{\sigma}_m(v_m) \end{bmatrix} \tag{20}$$

is the derivative of vector $\boldsymbol{\sigma}$ *with respect to the input signals* $v_i,\ i=1,\ldots,m$*, where* $[v_1,\ldots,v_m]^T = \mathbf{V}^T\boldsymbol{\zeta}$ *and m denotes the number of neurons in the hidden layer.*

Proof. Using the Taylor series expansion of $\boldsymbol{\sigma}(\mathbf{V}^{*T}\boldsymbol{\zeta})$, it yields

$$\boldsymbol{\sigma}(\mathbf{V}^{*T}\boldsymbol{\zeta}) = \boldsymbol{\sigma}(\mathbf{V}^T\boldsymbol{\zeta} + \tilde{\mathbf{V}}^T\boldsymbol{\zeta}) = \boldsymbol{\sigma}(\mathbf{V}^T\boldsymbol{\zeta}) + \dot{\boldsymbol{\sigma}}(\mathbf{V}^T\boldsymbol{\zeta})\tilde{\mathbf{V}}^T\boldsymbol{\zeta} + \mathbf{O}(\cdot) \tag{21}$$

where $\mathbf{O}(\cdot) \in R^m$ denotes the vector associated with high order terms. Note that the nonlinear activation function of neurons in the hidden layer of MLP $\sigma_i(v_i)$ is a sigmoid type (e.g. the logistic function $\sigma_i(v_i) = 1/(1+e^{-\alpha v_i})$ or the hyperbolic tangent function $\sigma_i(v_i) = \tanh(\alpha v_i)$ with $\alpha > 0$). Hence, $|\sigma_i| \le 1$ and $|\partial\sigma_i(v_i)/v_i| \le \alpha$. Consequently, for both cases, $\|\boldsymbol{\sigma}\| \le \sqrt{m}$ and $\|\dot{\boldsymbol{\sigma}}\| \le \alpha$. Using these inequalities and (21), it can be shown that $\|\mathbf{O}(\cdot)\|$ is also bounded

$$\|\mathbf{O}(\cdot)\| = \left\|\boldsymbol{\sigma}(\mathbf{V}^{*T}\boldsymbol{\zeta}) - \boldsymbol{\sigma}(\mathbf{V}^T\boldsymbol{\zeta}) - \dot{\boldsymbol{\sigma}}(\mathbf{V}^T\boldsymbol{\zeta})\tilde{\mathbf{V}}^T\boldsymbol{\zeta}\right\| \le 2\sqrt{m} + \alpha\left\|\tilde{\mathbf{V}}\right\|_F \|\boldsymbol{\zeta}\| \tag{22}$$

From (21) the approximation error is evaluated

$$\begin{aligned}
\Delta - u_{ad} &= \mathbf{w}^{*T}\boldsymbol{\sigma}(\mathbf{V}^{*T}\boldsymbol{\zeta}) + \varepsilon_1 - \mathbf{w}^T\boldsymbol{\sigma}(\mathbf{V}^T\boldsymbol{\zeta}) \\
&= \left(\mathbf{w}^T + \tilde{\mathbf{w}}^T\right)\left(\boldsymbol{\sigma}(\mathbf{V}^T\boldsymbol{\zeta}) + \dot{\boldsymbol{\sigma}}(\mathbf{V}^T\boldsymbol{\zeta})\left(\mathbf{V}^{*T} - \mathbf{V}^T\right)\boldsymbol{\zeta} + \mathbf{O}(\cdot)\right) + \varepsilon_1 - \mathbf{w}^T\boldsymbol{\sigma}(\mathbf{V}^T\boldsymbol{\zeta}) \\
&= \tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma} - \dot{\boldsymbol{\sigma}}\mathbf{V}^T\boldsymbol{\zeta}\right) + \mathbf{w}^T\dot{\boldsymbol{\sigma}}\tilde{\mathbf{V}}^T\boldsymbol{\zeta} + \tilde{\mathbf{w}}^T\dot{\boldsymbol{\sigma}}\mathbf{V}^{*T}\boldsymbol{\zeta} + \mathbf{w}^{*T}\mathbf{O}(\cdot) + \varepsilon_1 \\
&= \tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma} - \dot{\boldsymbol{\sigma}}\mathbf{V}^T\boldsymbol{\zeta}\right) + \mathrm{tr}\left(\tilde{\mathbf{V}}^T\boldsymbol{\zeta}\mathbf{w}^T\dot{\boldsymbol{\sigma}}\right) + \delta(t)
\end{aligned}$$

where

$$\delta(t) \triangleq \tilde{\mathbf{w}}^T\dot{\boldsymbol{\sigma}}\mathbf{V}^{*T}\boldsymbol{\zeta} + \mathbf{w}^{*T}\mathbf{O}(\cdot) + \varepsilon_1$$

Now, using (17), (22), $\|\dot{\boldsymbol{\sigma}}\| \le \alpha$ and the fact that $\|\mathbf{AB}\|_F \le \|\mathbf{A}\|_F\|\mathbf{B}\|$, a bound for $\delta(t)$ is obtained

$$\begin{aligned}
|\delta(t)| &\le \|\tilde{\mathbf{w}}\|\|\dot{\boldsymbol{\sigma}}\|\left\|\mathbf{V}^*\right\|_F\|\boldsymbol{\zeta}\| + \left\|\mathbf{w}^*\right\|\|\mathbf{O}(\cdot)\| + \varepsilon_{1M} \\
&\le \alpha M_V\|\tilde{\mathbf{w}}\|\|\boldsymbol{\zeta}\| + M_w\left(2\sqrt{m} + \alpha\left\|\tilde{\mathbf{V}}\right\|_F\|\boldsymbol{\zeta}\|\right) + \varepsilon_{1M} \\
&= \varepsilon_{1M} + 2\sqrt{m}M_w + \alpha M_w\left\|\tilde{\mathbf{V}}\right\|_F\|\boldsymbol{\zeta}\| + \alpha M_V\|\tilde{\mathbf{w}}\|\|\boldsymbol{\zeta}\|
\end{aligned}$$

□

The adaptation rules for the weights of the neuro-adaptive control part u_{ad}, defined in (18), is proposed as

$$\left.\begin{aligned}\dot{\mathbf{w}} &= \gamma_{\mathrm{w}}\left(\rho_1\left(\boldsymbol{\sigma}-\dot{\boldsymbol{\sigma}}\mathbf{V}^T\boldsymbol{\zeta}\right)-k_w\mathbf{w}\right)\\ \dot{\mathbf{V}} &= \gamma_{\mathrm{V}}\left(\rho_1\boldsymbol{\zeta}\mathbf{w}^T\dot{\boldsymbol{\sigma}}-k_V\mathbf{V}\right)\end{aligned}\right\} \tag{23}$$

where ρ_1 is the same as in (8), γ_{w} and γ_{V} are learning coefficients, and k_w and k_V are σ-modification gains.

Note that in the case of noise-corrupted measurements, it is beneficial to use a data history as large as possible in input vector of NN, $\boldsymbol{\zeta}$, because the variance of the estimated parameters of NN (i.e. weights **V**, **w**) increases proportionally to the noise variance and to the inverse of size of covariance matrix of input data.

Lemma 1 and its proof differ from that which presented by Lewis *et al.* [23, 24]. In this paper, it is assumed that the modelling error term $\delta(t)$ is bounded with respect to the inputs of the NN, rather than with respect to the filtered tracking error.

Remark 2: As it is shown in Section 4, the stability analysis relies on an extension of the Lyapunov theory. The derivate of this Lyapunov function is negative outside a compact set. In this case, to avoid any persistent excitation condition on the NN inputs and to guarantee the boundedness of $\tilde{\mathbf{w}}$ and $\tilde{\mathbf{V}}$, the σ-modification terms are considered in the adaptation rules [23, 25].

Using (16), (17), (18), and $\|\boldsymbol{\sigma}\| \le \sqrt{m}$, the following conservative upper bound of the approximation error is obtained

$$\begin{aligned}\left|\Delta(\mathbf{z},\boldsymbol{\eta},u)-u_{ad}\right| &= \left|\mathbf{w}^{*T}\boldsymbol{\sigma}\left(\mathbf{V}^{*T}\boldsymbol{\zeta}\right)+\varepsilon_1-\mathbf{w}^T\boldsymbol{\sigma}\left(\mathbf{V}^T\boldsymbol{\zeta}\right)\right|\\ &\le \left|\mathbf{w}^{*T}\boldsymbol{\sigma}\left(\mathbf{V}^{*T}\boldsymbol{\zeta}\right)\right|+\left|\mathbf{w}^T\boldsymbol{\sigma}\left(\mathbf{V}^T\boldsymbol{\zeta}\right)\right|+\left|\varepsilon_1\right|\\ &\le \left\|\mathbf{w}^*\right\|\left\|\boldsymbol{\sigma}\right\|+\left\|\mathbf{w}\right\|\left\|\boldsymbol{\sigma}\right\|+\left|\varepsilon_1\right|\\ &\le 2\sqrt{m}\,M_W+\left|\varepsilon_1\right|\end{aligned} \tag{24}$$

3.4 Adaptive robustifying control design

The neuro-adaptive control part u_{ad}, with adaptation rules (23), may not completely eliminate the matched uncertainty since the approximation error $\delta(t)$ influences the system, In order to compensate this error, an adaptive robustifying control part u_R is proposed. Using (20), the upper bound of this approximation error can be calculated as

$$
\begin{aligned}
|\delta| &\le \left(\varepsilon_{1M} + 2\sqrt{m}M_{\mathrm{w}}\right) + \alpha M_{\mathrm{V}} \left\|\mathbf{w}^* - \mathbf{w}\right\|\|\zeta\| + \alpha M_{\mathrm{w}} \left\|\mathbf{V}^* - \mathbf{V}\right\|_F \|\zeta\| \\
&\le \left(\varepsilon_{1M} + 2\sqrt{m}M_{\mathrm{w}}\right) + \alpha M_{\mathrm{V}} \left\|\mathbf{w}^*\right\|\|\zeta\| + \alpha M_{\mathrm{V}} \|\mathbf{w}\|\|\zeta\| + \alpha M_{\mathrm{w}} \left\|\mathbf{V}^*\right\|_{\mathrm{F}} \|\zeta\| + \alpha M_{\mathrm{w}} \|\mathbf{V}\|_{\mathrm{F}} \|\zeta\| \\
&\le \left(\varepsilon_{1M} + 2\sqrt{m}M_{\mathrm{w}}\right) + \alpha M_{\mathrm{V}} M_{\mathrm{w}} \|\zeta\| + \alpha M_{\mathrm{V}} \|\mathbf{w}\|\|\zeta\| + \alpha M_{\mathrm{w}} M_{\mathrm{V}} \|\zeta\| + \alpha M_{\mathrm{w}} \|\mathbf{V}\|_{\mathrm{F}} \|\zeta\| \\
&\le \varphi^*\left(1 + \|\zeta\|\left(1 + \|\mathbf{V}\|_{\mathrm{F}} + \|\mathbf{w}\|\right)\right) = \varphi^* \chi
\end{aligned}
\tag{25}
$$

where $\varphi^* = \max\left\{\varepsilon_{1M} + 2\sqrt{m}M_{\mathrm{w}}, \alpha M_{\mathrm{w}}, \alpha M_{\mathrm{V}}, 2\alpha M_{\mathrm{V}} M_{\mathrm{w}}\right\}$ and $\chi \triangleq 1 + \|\zeta\|\left(1 + \|\mathbf{V}\|_{\mathrm{F}} + \|\mathbf{w}\|\right)$. Hence, $\delta(t)$ is limited to the multiplication of the known function χ and an unknown parameter φ^*. Therefore, the following adaptive robustifying control part is introduced

$$
u_R = \chi\, \varphi \tanh\left(\rho_1 / \mu_R\right) \tag{26}
$$

with the following adaptation rule

$$
\dot{\varphi} = \gamma_\varphi\, \chi\, |\rho_1| \tag{27}
$$

where μ_R is a small positive constant, γ_φ is the constant learning rate and φ is an estimate of the unknown parameter φ^*. The adaptation rule for φ is derived in Section 4 using the Lyapunov direct method. Note that, in order to eliminate the chattering phenomenon, the continuous function $\tanh(\cdot)$ is used here instead of the conventional discontinuous function $\mathrm{sign}(\cdot)$; nevertheless, this increases the ultimate error bound. Because of the universal approximation property of NNs, the approximation error is bounded. Hence, it is always possible to find a positive constant U_M such that

$$
|u_R| \le U_M \tag{28}
$$

3.5 Observer design

For realisation of weight adaptation laws, given in (23) and (27), (i.e. dependent only on the measurable system output), the linear state estimator

$$
\dot{\hat{\xi}} = \mathbf{A}\hat{\xi} + \mathbf{b}u_L + \mathbf{k}_o\left(e - \mathbf{c}\hat{\xi}\right) \tag{29}
$$

is proposed where $\mathbf{b}$ and $\mathbf{c}$ are the same as in (6) and the observer gain $\mathbf{k}_o = [k_1 \;\cdots\; k_n]^T$ is selected such that $\mathbf{A} - \mathbf{k}_o\mathbf{c}$ is stable. Moreover, the stability of $\mathbf{A} - \mathbf{k}_o\mathbf{c}$ assures the existence of the symmetric positive-definite solution $\mathbf{P}_2$ of the algebraic Riccati equation

$$
\mathbf{P}_2\left(\mathbf{A} - \mathbf{k}_o\mathbf{c}\right) + \left(\mathbf{A} - \mathbf{k}_o\mathbf{c}\right)^T \mathbf{P}_2 = -\mathbf{Q}_2 - \mathbf{c}^T\mathbf{k}_o^T\mathbf{P}_1\mathbf{Q}_1^{-1}\mathbf{P}_1\mathbf{k}_o\mathbf{c} \tag{30}
$$

where $\mathbf{Q}_2$ is an arbitrary symmetric positive-definite matrix.

The observer (29) is included to the nonlinear system (6), as depicted in Fig. 1. Define the state estimation error as $\tilde{\boldsymbol{\xi}} \triangleq \hat{\boldsymbol{\xi}} - \boldsymbol{\xi}$ and

$$\mathbf{E} \triangleq \begin{bmatrix} \boldsymbol{\xi}^T & \tilde{\boldsymbol{\xi}}^T \end{bmatrix}^T \tag{31}$$

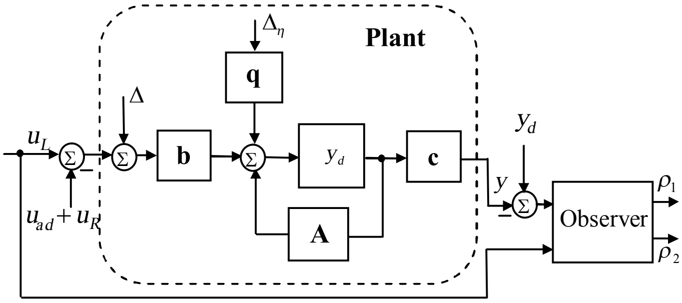

Fig. 1. Block diagram of the augmented plant

Then, the augmented system dynamics can be described as

$$\dot{\mathbf{E}} = \underbrace{\begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k}_c & -\mathbf{b}\mathbf{k}_c \\ 0 & \mathbf{A} - \mathbf{k}_o\mathbf{c} \end{bmatrix}}_{\triangleq \mathbf{A}_0} \mathbf{E} + \underbrace{\begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}}_{\triangleq \mathbf{b}_0} \left(u_L + \mathbf{k}_c \hat{\boldsymbol{\xi}} + \beta \right) + \underbrace{\begin{bmatrix} \mathbf{q} \\ 0 \end{bmatrix}}_{\triangleq \mathbf{q}_0} \gamma - \underbrace{\begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}}_{\triangleq \mathbf{b}_1} \beta - \underbrace{\begin{bmatrix} 0 \\ \mathbf{q} \end{bmatrix}}_{\triangleq \mathbf{q}_1} \gamma$$

where $\beta \triangleq \Delta - u_{ad} - u_R$ and $\gamma \triangleq y^* + \Delta_{\boldsymbol{\eta}}$.

Therefore, the augmented system dynamic can be expressed as

$$\dot{\mathbf{E}} = \mathbf{A}_0 \mathbf{E} + \mathbf{b}_0 \left(u_L + \mathbf{k}_c \hat{\boldsymbol{\xi}} + \beta \right) + \mathbf{q}_0 \gamma - \mathbf{b}_1 \beta - \mathbf{q}_1 \gamma \tag{32}$$

where

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k}_c & -\mathbf{b}\mathbf{k}_c \\ 0 & \mathbf{A} - \mathbf{k}_o\mathbf{c} \end{bmatrix},\ \mathbf{b}_0 = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix},\ \mathbf{q}_0 = \begin{bmatrix} \mathbf{q} \\ 0 \end{bmatrix},\ \mathbf{b}_1 = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix},\ \mathbf{q}_1 = \begin{bmatrix} 0 \\ \mathbf{q} \end{bmatrix},\ \beta = \Delta - u_{ad} - u_R,\ \gamma = y^* + \Delta_{\boldsymbol{\eta}}$$

So the available output signals are introduced as

$$\left. \begin{aligned} \rho_1 &= \mathbf{k}_c \hat{\boldsymbol{\xi}} = \begin{bmatrix} \mathbf{k}_c & \mathbf{k}_c \end{bmatrix} \mathbf{E} \\ \rho_2 &= \mathbf{q}^T \mathbf{P}_1 \hat{\boldsymbol{\xi}} = \begin{bmatrix} \mathbf{q}^T \mathbf{P}_1 & \mathbf{q}^T \mathbf{P}_1 \end{bmatrix} \mathbf{E} \end{aligned} \right\}. \tag{33}$$

Remark 3: Let $\mathbf{b}_2 = \begin{bmatrix} \mathbf{b}^T & -\mathbf{b}^T \end{bmatrix}^T$ and $\mathbf{q}_2 = \begin{bmatrix} \mathbf{q}^T & -\mathbf{q}^T \end{bmatrix}^T$. If only the linear part of the control law, defined as in (8), is applied to the system, then, since u_L and y_d are functions of **E**, the closed-loop form of the system (32) can be considered as

$$\dot{\mathbf{E}} = \mathbf{A}_0 \mathbf{E} + \underbrace{\begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}}_{\mathbf{b}_2} \bar{\Delta}(\mathbf{E}) + \underbrace{\begin{bmatrix} \mathbf{q} \\ -\mathbf{q} \end{bmatrix}}_{\mathbf{q}_2} \bar{\gamma}(\mathbf{E}), \tag{34}$$

where $\bar{\Delta}(\mathbf{E})$ and $\bar{\gamma}(\mathbf{E})$ are continuous Lipschitz functions and act as vanishing perturbations because of $\bar{\Delta}(\mathbf{0})=0$ and $\bar{\gamma}(\mathbf{0})=0$. This can be concluded from the conditions $f(\mathbf{0},\mathbf{0},0)=0$ and $v(\mathbf{0},\mathbf{0})=\mathbf{0}$ as defined in (1) [17]. In the following Lemma, it is shown that by designing suitable feedback gains $\mathbf{k}_c$ and $\mathbf{k}_o$, the boundedness of states is achieved in the presence of these perturbations.

Lemma 2: *Consider the system (32) and* $\mathbf{Q}=-\left(\mathbf{P}\mathbf{A}_0+\mathbf{A}_0^T\mathbf{P}\right)$ where

$$\mathbf{P}=\begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_1\\ \mathbf{P}_1 & \mathbf{P}_1+\mathbf{P}_2\end{bmatrix}$$

If only the linear controller (8) is applied and the condition

$$\lambda_{\min}(\mathbf{Q})>2\left(\gamma_{11}\left\|\mathbf{b}_2\right\|+\gamma_{22}\left\|\mathbf{q}_2\right\|\right)\lambda_{\max}(\mathbf{P})+2$$

is satisfied then the system states are bounded in the presence of the perturbation signals $\bar{\Delta}(\mathbf{E})$ *and* $\bar{\gamma}(\mathbf{E})$.

Proof. Define the following conic bounds for perturbations [17]

$$\begin{cases}\left|\bar{\Delta}(\mathbf{E})\right|\le\gamma_{12}+\gamma_{11}\left\|\mathbf{E}\right\|\\ \left|\bar{\gamma}(\mathbf{E})\right|\le\gamma_{21}+\gamma_{22}\left\|\mathbf{E}\right\|,\end{cases}\tag{35}$$

where $\gamma_{11},\gamma_{12},\gamma_{21}$ and γ_{22} are appropriate nonnegative real constants. Now consider the quadratic Lyapunov function as

$$L_1=\frac{1}{2}\mathbf{E}^T\mathbf{P}\mathbf{E}.$$

Using (34) and bounds defined in (35), the time-derivate of L_1 becomes

$$\begin{aligned}\dot{L}_1&=-\frac{1}{2}\mathbf{E}^T\mathbf{Q}\mathbf{E}+\mathbf{E}^T\mathbf{P}\mathbf{b}_2\bar{\Delta}+\mathbf{E}^T\mathbf{P}\mathbf{q}_2\bar{\gamma}\\ &\le-\frac{1}{2}\lambda_{\min}(\mathbf{Q})\left\|\mathbf{E}\right\|^2+\left\|\mathbf{E}^T\mathbf{P}\mathbf{b}_2\bar{\Delta}\right\|+\left\|\mathbf{E}^T\mathbf{P}\mathbf{q}_2\bar{\gamma}\right\|\\ &\le-\left(\frac{1}{2}\lambda_{\min}(\mathbf{Q})-\gamma_{11}\left\|\mathbf{b}_2\right\|\lambda_{\max}(\mathbf{P})-\gamma_{22}\left\|\mathbf{q}_2\right\|\lambda_{\max}(\mathbf{P})-1\right)\left\|\mathbf{E}\right\|^2+\frac{1}{2}\left(\gamma_{12}^2\left\|\mathbf{b}_2\right\|^2+\gamma_{21}^2\left\|\mathbf{q}_2\right\|^2\right)\lambda_{\max}^2(\mathbf{P}),\end{aligned}\tag{36}$$

since $\mathbf{Q}=-\left(\mathbf{P}\mathbf{A}_0+\mathbf{A}_0^T\mathbf{P}\right)$,

$$\begin{aligned}\left\|\mathbf{E}^T\mathbf{P}\mathbf{b}_2\bar{\Delta}\right\| &\le \|\mathbf{E}\|\|\mathbf{b}_2\|\|\bar{\Delta}\|\lambda_{\max}(\mathbf{P}) \\ &\le \gamma_{11}\|\mathbf{E}\|^2\|\mathbf{b}_2\|\lambda_{\max}(\mathbf{P}) + \gamma_{12}\|\mathbf{E}\|\|\mathbf{b}_2\|\lambda_{\max}(\mathbf{P}) \\ &\le \gamma_{11}\|\mathbf{E}\|^2\|\mathbf{b}_2\|\lambda_{\max}(\mathbf{P}) + \frac{1}{2}\left(\|\mathbf{E}\|^2 + \gamma_{12}^2\|\mathbf{b}_2\|^2\lambda_{\max}^2(\mathbf{P})\right)\end{aligned}$$

and

$$\begin{aligned}\left\|\mathbf{E}^T\mathbf{P}\mathbf{q}_2\bar{\gamma}\right\| &\le \|\mathbf{E}\|\|\mathbf{q}_2\|\|\bar{\gamma}\|\lambda_{\max}(\mathbf{P}) \\ &\le \gamma_{22}\|\mathbf{E}\|^2\|\mathbf{q}_2\|\lambda_{\max}(\mathbf{P}) + \gamma_{21}\|\mathbf{E}\|\|\mathbf{q}_2\|\lambda_{\max}(\mathbf{P}) \\ &\le \gamma_{22}\|\mathbf{E}\|^2\|\mathbf{q}_2\|\lambda_{\max}(\mathbf{P}) + \frac{1}{2}\left(\|\mathbf{E}\|^2 + \gamma_{21}^2\|\mathbf{q}_2\|^2\lambda_{\max}^2(\mathbf{P})\right).\end{aligned}$$

Therefore, if the following condition is satisfied

$$\lambda_{\min}(\mathbf{Q}) > 2\left(\gamma_{11}\|\mathbf{b}_2\| + \gamma_{22}\|\mathbf{q}_2\|\right)\lambda_{\max}(\mathbf{P}) + 2\,, \tag{37}$$

then the boundedness of states is assured while only the linear control is applied. □

Alternatively, a new sufficient condition for boundedness of the system states, similar to the condition (37) may be obtained. To achieve this, it is assumed that a matrix $\mathbf{S}$ is similar to the nonsigular matrix $\mathbf{A}_0$ such that $\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}_0\mathbf{S}$ is a diagonal matrix. In particular, the entries of $\mathbf{\Lambda}$ may be selected the same as the eigenvalues of $\mathbf{A}_0$. Define new state variables as $\bar{\mathbf{E}} \triangleq \mathbf{S}^{-1}\mathbf{E}$. By replacing $\mathbf{E}$ with $\bar{\mathbf{E}}$ in (34) and (35) and considering a similar procedure for deriving $\dot{L}_1$ as stated in (36), the following condition is obtained

$$\lambda_{\min}\left(\bar{\mathbf{Q}}\right) > 2\left(\gamma_{11}\|\bar{\mathbf{b}}_2\| + \gamma_{22}\|\bar{\mathbf{q}}_2\|\right)\|\mathbf{S}\|\lambda_{\max}(\bar{\mathbf{P}}) + 2 \tag{38}$$

where $\bar{\mathbf{Q}} = \mathbf{S}^T\mathbf{Q}\mathbf{S}$, $\bar{\mathbf{P}} = \mathbf{S}^T\mathbf{P}\mathbf{S}$, $\bar{\mathbf{b}}_2 = \mathbf{S}^{-1}\mathbf{b}_2$ and $\bar{\mathbf{q}}_2 = \mathbf{S}^{-1}\mathbf{q}_2$. From (6) and (32), $\|\mathbf{b}_2\| = \sqrt{2}b$, $\|\mathbf{q}_2\| = \sqrt{2}$. Moreover, $\bar{\mathbf{P}} = \bar{\mathbf{P}}^T$. Hence,

$$\begin{aligned}\left(\gamma_{11}\|\bar{\mathbf{b}}_2\| + \gamma_{22}\|\bar{\mathbf{q}}_2\|\right)\|\mathbf{S}\|\|\bar{\mathbf{P}}\| &\le \sqrt{2}\left(b\gamma_{11} + \gamma_{22}\right)\|\mathbf{S}^{-1}\|\|\mathbf{S}\|\lambda_{\max}(\bar{\mathbf{P}}) \\ &= \sqrt{2}\left(b\gamma_{11} + \gamma_{22}\right)\|\mathbf{S}^{-1}\|\|\mathbf{S}\|\lambda_{\max}(\bar{\mathbf{P}})\end{aligned} \tag{39}$$

On the other hand, since $\mathbf{A}_0$ is a stable matrix, the real parts of all its eigenvalues are within the left half-plane. So $(\mathbf{\Lambda} + \mathbf{\Lambda}^T)/2 < 0$, particularly $\lambda_{\max}(\mathbf{\Lambda} + \mathbf{\Lambda}^T) < 0$. In addition,

$$\lambda_{\max}(\bar{\mathbf{P}}) \le \frac{\lambda_{\max}(\bar{\mathbf{Q}})}{\left|\lambda_{\max}(\mathbf{\Lambda} + \mathbf{\Lambda}^T)\right|}. \tag{40}$$

[26]. Using (39) and (40) it is concluded that (38) holds if the following condition on the maximum and minimum eigenvalues of the weighting matrix $\bar{\mathbf{Q}}$ is satisfied

$$\lambda_{\min}(\bar{\mathbf{Q}}) > 2\left(1 + \frac{\sqrt{2}\left(b\gamma_{11} + \gamma_{22}\right)\|\mathbf{S}^{-1}\|\|\mathbf{S}\|\lambda_{\max}(\bar{\mathbf{Q}})}{\left|\lambda_{\max}(\mathbf{\Lambda} + \mathbf{\Lambda}^T)\right|}\right) \tag{41}$$

In order to satisfy this condition, eigenvalues of $\mathbf{A}_0$ or equivalently $\mathbf{\Lambda}$, which are associated with the linear part of the controller, should be appropriately selected. This can be achieved by designing suitable feedback gains $\mathbf{k}_c$ and $\mathbf{k}_o$.

Therefore, the closed-loop system remains stable until the NN (i.e. the adaptive part u_{ad}) and y_d begin to learn. The weights are tuned online as the system tracks the desired trajectory. As the NN and y_d learn the matched and the unmatched uncertainties, respectively, the tracking performance improves and the ultimate error bound decreases. It is important to note that at the commencing of the learning process to achieve the system stability, the existence of the compact set Ω_ς, as defined in (16), is guaranteed.

3.6 Reference signal construction

The reference signal y_d (see Fig. 1) is designed to cancel out the unmatched uncertainty $\Delta_{\boldsymbol{\eta}}$. Using the error $e \triangleq y_d - y$, the upper bound of the modelling error, defined in (5), can be represented as

$$\left|\boldsymbol{\Delta}_{\boldsymbol{\eta}}(\mathbf{z},\boldsymbol{\eta})\right| \le c_0^* + c_1\left(\|\mathbf{e}\| + \|\mathbf{y}_d\| + \left|y_d^{(r)}\right|\right) + c_2^*\|\boldsymbol{\eta}\| \tag{42}$$

On the other hand, from (3) and (4) the following bounds can be derived

$$\|\mathbf{y}_d\| + \left|y_d^{(r)}\right| = \sqrt{\sum_{i=0}^{r-1}\left(y_d^{(i)}\right)^2} + \left|y_d^{(r)}\right| \le \sum_{i=0}^{r}\left|y_d^{(i)}\right| \tag{43}$$

$$\left|y^*\right| = \left|\sum_{i=0}^{r-1} g_{i+1} y_d^{(i)}\right| \le \sum_{i=0}^{r-1}\left|g_{i+1}\right|\left|y_d^{(i)}\right| \tag{44}$$

Then

$$\|\mathbf{y}_d\| + \left|y_d^{(r)}\right| \le \left|y^*\right| - \left|\sum_{i=0}^{r-1} g_{i+1} y_d^{(i)}\right| + \sum_{i=0}^{r}\left|y_d^{(i)}\right| \le \left|y^*\right| + c_3^* p \tag{45}$$

where $p \le 1$ is a nonnegative real number and c_3^* is defined in (5). Substituting (45) into (42) yields

$$\left|\boldsymbol{\Delta}_{\boldsymbol{\eta}}(\mathbf{z},\boldsymbol{\eta})\right| \le c_4^* + c_1\left|y^*\right| + c_5^*\|\xi\| \tag{46}$$

where $c_5^* = c_1 + c_2^*$ and $c_4^* = c_0^* + c_1 c_3^* p$. Define $\boldsymbol{\lambda}^* \triangleq \begin{bmatrix} c_4^* & c_5^* \end{bmatrix}^T$. Then

$$\left|\boldsymbol{\Delta}_{\boldsymbol{\eta}}(\mathbf{z},\boldsymbol{\eta})\right| \le c_1\left|y^*\right| + \boldsymbol{\lambda}^{*T}\begin{bmatrix} 1 & \|\xi\| \end{bmatrix}^T \tag{47}$$

Let $\boldsymbol{\lambda}$ be an estimate of the unknown parameter $\boldsymbol{\lambda}^*$. An adaptive reference signal is proposed as

$$y_d = \frac{1}{D(s)} y^* = \frac{1}{D(s)} \left(-\frac{\boldsymbol{\lambda}^T \begin{bmatrix} 1 & \|\hat{\boldsymbol{\xi}}\| \end{bmatrix}^T}{1-c_1} \tanh\left(\frac{\rho_2}{\mu_y}\right) \right) \tag{48}$$

with the following adaptation rule

$$\dot{\boldsymbol{\lambda}} = \begin{bmatrix} \dot{c}_4 & \dot{c}_5 \end{bmatrix}^T = \Gamma_\lambda \begin{bmatrix} 1 & \|\hat{\boldsymbol{\xi}}\| \end{bmatrix}^T |\rho_2| \tag{49}$$

where μ_y is a positive constant, Γ_λ is the learning coefficient matrix and

$$D(s) = g_r s^{r-1} + g_{r-1} s^{r-2} + \cdots + g_1$$

is stable polynomial in which $g_i \ (i = 1, \ldots, r)$ were defined in Section 3.

Remark 4: In practice, small positive numbers are selected as the initial values of $\begin{bmatrix} c_4 & c_5 \end{bmatrix}$. Then, according to (49) these gains increase and approach to $\begin{bmatrix} c_4^* & c_5^* \end{bmatrix}$. Hence, always $c_4 \le c_4^*$ and $c_5 \le c_5^*$. Using the approximation error $\gamma = y^* + \Delta_{\boldsymbol{\eta}}$, and equations (46) and (48), the following bound can be derived

$$|\gamma| \le c_4^* (1+d) + c_5^* d \|\tilde{\boldsymbol{\xi}}\| + c_5^* (1+d) \|\boldsymbol{\xi}\| \tag{50}$$

where $d = \dfrac{1+c_1}{1-c_1}$. Substituting (31) into (50) yields

$$|\gamma| \le \alpha_0 + \alpha_1 \|\mathbf{E}\| \tag{51}$$

where $\alpha_0 = c_4^*(1+d)$ and $\alpha_1 = c_5^*(1+2d)$.

4 Stability analysis

This section presents stability analysis of the proposed control law in the closed-loop system. Using an extension of the Lyapunov theory, it is shown in the following theorem that the error trajectories $\mathbf{E}$, $\tilde{\mathbf{w}}$ and $\tilde{\mathbf{V}}$ are ultimately bounded.

Theorem 1: *Consider the linear controller (8), the neuro-adaptive control part u_{ad} in (18) with the adaptation rules (23), the robustifying control part u_R in (26), and the reference signal y_d in (48). Then, the error signals $\mathbf{E}$, $\tilde{\mathbf{w}}$, and $\tilde{\mathbf{V}}$ in the closed-loop system are uniformly ultimately bounded.*

Proof: Define the Lyapunov function

$$L = \frac{1}{2}\hat{\boldsymbol{\xi}}^T \mathbf{P}_1 \hat{\boldsymbol{\xi}} + \frac{1}{2}\tilde{\boldsymbol{\xi}}^T \mathbf{P}_2 \tilde{\boldsymbol{\xi}} + \frac{1}{2\gamma_{\mathbf{w}}}\|\tilde{\mathbf{w}}\|^2 + \frac{1}{2\gamma_V}\|\tilde{\mathbf{V}}\|_F^2 + \frac{1}{2\gamma_\varphi}|\tilde{\varphi}|^2 + \frac{1}{2}\tilde{\boldsymbol{\lambda}}^T \Gamma_\lambda^{-1} \tilde{\boldsymbol{\lambda}}$$

where $\tilde{\varphi} \triangleq \varphi^* - \varphi$ and $\tilde{\boldsymbol{\lambda}} \triangleq \boldsymbol{\lambda}^* - \boldsymbol{\lambda}$, in which φ^* and $\boldsymbol{\lambda}^*$ are the ideal gains of their corresponding estimated values φ and $\boldsymbol{\lambda}$, respectively. Using (31), this Lyapunov function can be represented as

$$L = \frac{1}{2}\mathbf{E}^T\mathbf{P}\mathbf{E} + \frac{1}{2\gamma_{\mathbf{w}}}\|\tilde{\mathbf{w}}\|^2 + \frac{1}{2\gamma_V}\|\tilde{\mathbf{V}}\|_F^2 + \frac{1}{2\gamma_\varphi}|\tilde{\varphi}|^2 + \frac{1}{2}\tilde{\boldsymbol{\lambda}}^T\Gamma_\lambda^{-1}\tilde{\boldsymbol{\lambda}} \tag{52}$$

where $\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}$. Recall that $\mathbf{w}^*$ and $\mathbf{V}^*$ are the ideal constant weights for the NN, defined in (17). Then, from (20) $\dot{\mathbf{w}} = -\dot{\tilde{\mathbf{w}}}$ and $\dot{\mathbf{V}} = -\dot{\tilde{\mathbf{V}}}$. Using (32), the time-derivative of (52) becomes

$$\begin{aligned}
\dot{L} = {} & \frac{1}{2}\mathbf{E}^{\mathrm{T}}\left(\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}\begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k}_c & -\mathbf{b}\mathbf{k}_c \\ 0 & \mathbf{A} - \mathbf{k}_o\mathbf{c} \end{bmatrix} + \begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k}_c & -\mathbf{b}\mathbf{k}_c \\ 0 & \mathbf{A} - \mathbf{k}_o\mathbf{c} \end{bmatrix}^T\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}\right)\mathbf{E} \\
& + \mathbf{E}^T\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}\mathbf{b}_0\left(\beta + u_L + \mathbf{k}_c\hat{\boldsymbol{\xi}}\right) + \mathbf{E}^T\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}\mathbf{q}_0\gamma \\
& - \mathbf{E}^T\mathbf{P}\mathbf{b}_1\beta - \mathbf{E}^T\mathbf{P}\mathbf{q}_1\gamma - \frac{1}{\gamma_w}\tilde{\mathbf{w}}^T\dot{\mathbf{w}} - \frac{1}{\gamma_V}\mathrm{tr}\left(\tilde{\mathbf{V}}^T\dot{\mathbf{V}}\right) - \frac{1}{\gamma_\varphi}\tilde{\varphi}\dot{\varphi} - \tilde{\boldsymbol{\lambda}}^T\Gamma_\lambda^{-1}\dot{\boldsymbol{\lambda}}
\end{aligned}$$

Using (9) and (33), yields

$$\mathbf{E}^T\mathbf{P}\mathbf{b}_0 = \mathbf{E}^T\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}\mathbf{b}_0 = \mathbf{E}^T\begin{bmatrix} \mathbf{P}_1\mathbf{b} \\ \mathbf{P}_1\mathbf{b} \end{bmatrix} = \mathbf{E}^T\begin{bmatrix} \mathbf{k}_c^T \\ \mathbf{k}_c^T \end{bmatrix} = \rho_1 \tag{53}$$

$$\mathbf{E}^T\mathbf{P}\mathbf{q}_0 = \mathbf{E}^T\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}\mathbf{q}_0 = \mathbf{E}^T\begin{bmatrix} \mathbf{P}_1\mathbf{q} \\ \mathbf{P}_1\mathbf{q} \end{bmatrix} = \rho_2 . \tag{54}$$

Using (8), (10), (30), (53), and (54), $\dot{L}$ becomes

$$\dot{L} = -\frac{1}{2}\mathbf{E}^{\mathrm{T}}\mathbf{Q}\mathbf{E} + \rho_1\beta + \rho_2\gamma - \mathbf{E}^T\mathbf{P}\mathbf{b}_1\beta - \mathbf{E}^T\mathbf{P}\mathbf{q}_1\gamma - \frac{1}{\gamma_w}\tilde{\mathbf{w}}^T\dot{\mathbf{w}} - \frac{1}{\gamma_V}\mathrm{tr}\left(\tilde{\mathbf{V}}^T\dot{\mathbf{V}}\right) - \frac{1}{\gamma_\varphi}\tilde{\varphi}\dot{\varphi} - \tilde{\boldsymbol{\lambda}}^T\Gamma_\lambda^{-1}\dot{\boldsymbol{\lambda}}$$

where

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_1 + \mathbf{P}_1\mathbf{k}_o\mathbf{c} \\ \mathbf{Q}_1 + \mathbf{c}^T\mathbf{k}_o^T\mathbf{P}_1 & \mathbf{Q}_2 + \left(\mathbf{Q}_1 + \mathbf{c}^T\mathbf{k}_o^T\mathbf{P}_1\right)\mathbf{Q}_1^{-1}\left(\mathbf{Q}_1 + \mathbf{P}_1\mathbf{k}_o\mathbf{c}\right) \end{bmatrix}$$

Since $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are symmetric positive-definite matrices, $\mathbf{Q}$ is also a symmetric positive-definite matrix. Substituting $\beta = \tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma} - \dot{\boldsymbol{\sigma}}\mathbf{V}^T\boldsymbol{\zeta}\right) + \mathrm{tr}\left(\tilde{\mathbf{V}}^T\boldsymbol{\zeta}\mathbf{w}^T\dot{\boldsymbol{\sigma}}\right) + \delta - u_R$ from (19) and (32) into the last equation yields

$$\begin{aligned}
\dot{L} = {} & -\frac{1}{2}\mathbf{E}^{\mathrm{T}}\mathbf{Q}\mathbf{E} + \rho_1\left(\tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma} - \dot{\boldsymbol{\sigma}}\mathbf{V}^T\boldsymbol{\zeta}\right) + \mathrm{tr}\left(\tilde{\mathbf{V}}^T\boldsymbol{\zeta}\mathbf{w}^T\dot{\boldsymbol{\sigma}}\right)\right) - \frac{1}{\gamma_w}\tilde{\mathbf{w}}^T\dot{\mathbf{w}} - \frac{1}{\gamma_V}\mathrm{tr}\left(\tilde{\mathbf{V}}^T\dot{\mathbf{V}}\right) \\
& + \rho_1\left(\delta - u_R\right) - \frac{1}{\gamma_\varphi}\tilde{\varphi}\dot{\varphi} + \rho_2\left(y^* + \Delta_{\boldsymbol{\eta}}\right) - \tilde{\boldsymbol{\lambda}}^T\Gamma_\lambda^{-1}\dot{\boldsymbol{\lambda}} - \mathbf{E}^T\mathbf{P}\mathbf{b}_1\beta - \mathbf{E}^T\mathbf{P}\mathbf{q}_1\gamma,
\end{aligned} \tag{55}$$

Now from the bounds (25), (47), (51), the robustifying control part (26) and the reference signal (48), and considering the fact that $-x\tanh(x/\mu_x) \le -|x| + k\mu_x$ with $k = 0.2785$, and using (55), the time derivative of L satisfies the following inequality

$$\dot{L} \le -\frac{1}{2}\lambda_{\min}(\mathbf{Q})\|\mathbf{E}\|^2 + \tilde{\mathbf{w}}^T\left(\rho_1\boldsymbol{\psi} - k_w\mathbf{w} - \frac{1}{\gamma_w}\dot{\mathbf{w}}\right) + k_w\tilde{\mathbf{w}}^T\mathbf{w} + \mathrm{tr}\left(\tilde{\mathbf{V}}^T\left(\rho_1\boldsymbol{\Psi} - k_V\mathbf{V} - \frac{1}{\gamma_V}\dot{\mathbf{V}}\right)\right) + k_V\,\mathrm{tr}\left(\tilde{\mathbf{V}}^T\mathbf{V}\right)$$
$$+|\rho_1|\left(\varphi^* - \varphi\right)\chi - \frac{1}{\gamma_\varphi}\tilde{\varphi}\dot{\varphi} + k\mu_R U_M + |\rho_2|\boldsymbol{\lambda}^T\begin{bmatrix}1 & \|\hat{\boldsymbol{\xi}}\|\end{bmatrix}^T\left[\frac{c_1}{1-c_1} - \frac{1}{1-c_1}\right] + |\rho_2|\boldsymbol{\lambda}^{*T}\begin{bmatrix}1 & \|\hat{\boldsymbol{\xi}}\|\end{bmatrix}^T$$
$$-\tilde{\boldsymbol{\lambda}}^T\Gamma_\lambda^{-1}\dot{\boldsymbol{\lambda}} + \frac{\boldsymbol{\lambda}^{*T}k\mu_y}{1-c_1}\begin{bmatrix}1 & \|\hat{\boldsymbol{\xi}}\|\end{bmatrix}^T + c_5^*|\rho_2|\|\tilde{\boldsymbol{\xi}}\| + \|\mathbf{E}\|\|\mathbf{P}\mathbf{b}_1\|\beta_M + \|\mathbf{E}\|\|\mathbf{P}\mathbf{q}_1\|\left(\alpha_0 + \alpha_1\|\mathbf{E}\|\right),$$

where $\lambda_{\min}(\mathbf{Q})$ denotes the smallest eigenvalue of $\mathbf{Q}$ and $\beta_M \triangleq 2\sqrt{m}M_w + \varepsilon_{1M} + U_M$ is the upper bound of $|\beta|$ which can be derived using (24), (28) and (32). Let $\varepsilon \triangleq \dfrac{k\mu_y}{1-c_1}$. Using the inequalities $|\rho_2| \le \|\mathbf{P}\mathbf{q}_1\|\|\tilde{\boldsymbol{\xi}}\|$, $\|\hat{\boldsymbol{\xi}}\| \le \sqrt{2}\|\mathbf{E}\|$, $\|\tilde{\boldsymbol{\xi}}\| \le \|\mathbf{E}\|$ and applying the adaptation rules (23) yields

$$\dot{L} \le -\frac{1}{2}\lambda_{\min}(\mathbf{Q})\|\mathbf{E}\|^2 - k_w\|\tilde{\mathbf{w}}\|^2 + k_w M_w\|\tilde{\mathbf{w}}\| - k_V\|\tilde{\mathbf{V}}\|_F^2 + k_V M_V\|\tilde{\mathbf{V}}\|_F + \tilde{\varphi}\left(|\rho_1|\chi - \frac{1}{\gamma_\varphi}\dot{\varphi}\right) + k\mu_R U_M$$
$$+\tilde{\boldsymbol{\lambda}}^T\left(|\rho_2|\begin{bmatrix}1 & \|\hat{\boldsymbol{\xi}}\|\end{bmatrix}^T - \Gamma_\lambda^{-1}\dot{\boldsymbol{\lambda}}\right) + \varepsilon c_4^* + \sqrt{2}\varepsilon c_5^*\|\mathbf{E}\| + \sqrt{2}c_5^*\|\mathbf{P}\mathbf{q}_1\|\|\mathbf{E}\|^2$$
$$+\|\mathbf{E}\|\|\mathbf{P}\mathbf{b}_1\|\beta_M + \|\mathbf{E}\|\|\mathbf{P}\mathbf{q}_1\|\left(\alpha_0 + \alpha_1\|\mathbf{E}\|\right).$$

Using the adaptation rules (27) and (49), and completing the square terms gives

$$\dot{L} \le -A_E\|\mathbf{E}\|^2 - (k_w - 1)\|\tilde{\mathbf{w}}\|^2 - (k_V - 1)\|\tilde{\mathbf{V}}\|^2 + R \tag{56}$$

where

$$\left.\begin{aligned} A_E &\triangleq \left(\frac{1}{2}\lambda_{\min}(\mathbf{Q}) - \left(\alpha_1 + \sqrt{2}c_5^*\right)\|\mathbf{P}\mathbf{q}_1\| - 1\right) \\ R &\triangleq \frac{(k_w M_w)^2}{4} + \frac{(k_V M_V)^2}{4} + \frac{\left(\alpha_0\|\mathbf{P}\mathbf{q}_1\| + \beta_M\|\mathbf{P}\mathbf{b}_1\| + \varepsilon c_5^*\right)^2}{4} + \varepsilon c_4^* + k\mu_R U_M \end{aligned}\right\} \tag{57}$$

Select k_w, k_V, α_1 and c_5^* such that the following conditions are satisfied

$$\lambda_{\min}(\mathbf{Q}) > 2\left(\alpha_1 + \sqrt{2}c_5^*\right)\|\mathbf{P}\mathbf{q}_1\| + 2 \tag{58}$$
$$k_w > 1,\; k_V > 1$$

Define the following compact set around the origin

$$\Omega \triangleq \left\{\left(\mathbf{E}, \tilde{\mathbf{w}}, \tilde{\mathbf{V}}\right) \middle| A_E\|\mathbf{E}\|^2 + (k_w - 1)\|\tilde{\mathbf{w}}\|^2 + (k_V - 1)\|\tilde{\mathbf{V}}\|^2 \le R\right\}$$

Inequality (56) shows that when the errors are outside the compact set Ω, then $\dot{L} < 0$. Hence, according to the extension of the standard Lyapunov theorem [23, 27], the error trajectories $\mathbf{E}$, $\tilde{\mathbf{w}}$ and $\tilde{\mathbf{V}}$ are ultimately bounded. □

Note that condition (37) is obtained in the absence of the adaptive NN while the condition (58) is derived in the presence of the NN controller and the presence of the matched uncertainty does not affect to hold this condition. Therefore, the adaptive controller not only improves the system stability but also relaxes the stability condition.

Corollary: *The output y and the internal dynamics of the system (1) are bounded.*

Proof: (56) implies that $\dot{L}$ is strictly negative as long as $\mathbf{E}$ is outside the following compact set

$$\Omega_E \triangleq \left\{ \mathbf{E} \middle| \|\mathbf{E}\| \leq \sqrt{\frac{R}{A_E}} \right\} \tag{59}$$

Therefore, there exists a constant time T such that for $t > T$, the error $\mathbf{E}$ converges to Ω_E. This means that $\|\xi\| \leq \sqrt{R/A_E} = \varepsilon_E$, $\|\tilde{\xi}\| \leq \varepsilon_E$ and consequently $\|\hat{\xi}\| \leq \sqrt{2}\varepsilon_E$ and $\|e\| \leq \varepsilon_E$. Hence, (48) yields that $|y_d| \leq \varepsilon_y$ as $t \to \infty$, where $\varepsilon_y = \frac{c_4^* + \sqrt{2}c_5^*\varepsilon_E}{(1-c_1)} \tanh\left(\sqrt{2}\|\mathbf{P}_1\mathbf{q}\|\varepsilon_E / \mu_y \right)$. Therefore, the system output is stable and converges to a small bound, which is less than $\varepsilon_y + \varepsilon_E$ around the origin as $t \to \infty$. Moreover, the internal dynamics are bounded with a bound less than ε_E.

□

Remark 5: The equation (57) shows that the unmatched uncertainties, the NN reconstruction error embodied in the constants c_4^*, c_5^* and β_M increase the error bound. Note that, since $u_{ad} + u_R$ is designed to cancel out Δ, then the upper bound β_M, which is derived from (24) and (28), is selected very cautiously; however, in practice the real bound would be much smaller.

Fig. 2 shows the block diagram of the system with the proposed controller and observer in which TDL stands for tapped delay line such that the previous and current values are available to feed into the input of neural networks.

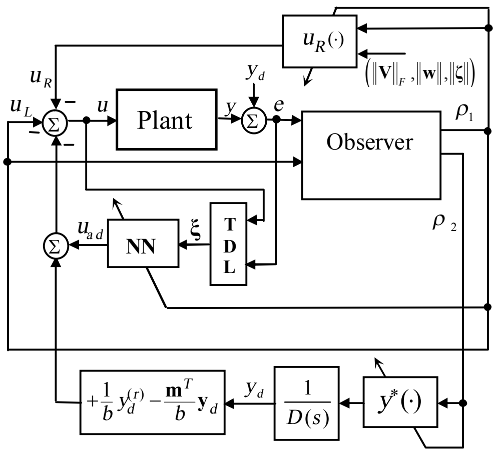

Fig. 2. Block diagram of the proposed controller

5 Example: Translational oscillator with a rotational actuator (TORA)

A TORA model is considered to illustrate the performance of the proposed controller [13, 15]. See Fig. 3. The dynamics of the system is governed by the following equations

$$(M+m)\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = -kx$$
$$(J+ml^2)\ddot{\theta} + ml\cos\theta\,\ddot{x} = \tau$$

where θ is the angle of rotation, x is the translational displacement, and τ is the control torque. The positive constants k, l, J, M, and m denote the spring stiffness, the radius of rotation, the moment of inertia, the mass of the cart, and the eccentric mass, respectively.

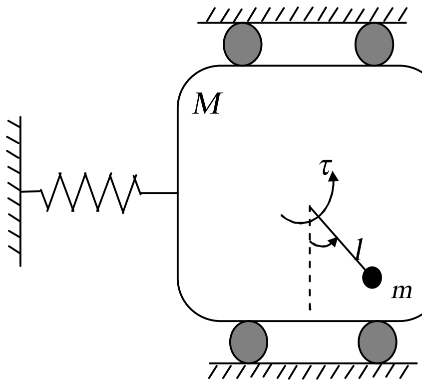

Fig. 3. A translational oscillator with a rotational actuator (TORA)

Define the states and the input variables as

$$\eta_1 = x + \frac{ml}{M+m}\sin\theta, \quad \eta_2 = \dot{x} + \frac{ml}{M+m}\dot{\theta}\cos\theta, \quad z_1 = \theta, \quad z_2 = \dot{\theta}, \quad u = \tau$$

In these coordinates, the system can be described by a set of equations in the form of (2) as

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = ka_1\phi^{-1}(z_1)\cos z_1\eta_1 - a_1^2 a_2\phi^{-1}(z_1)\sin z_1 \cos z_1 - m^2 l^2 \phi^{-1}(z_1) z_2^2 \sin z_1 \cos z_1 + (M+m)\phi^{-1}(z_1)\, u \\ \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = -a_2\eta_1 + a_3 \sin z_1 \end{cases}$$

where $\phi(z_1) = (M+m)(J+ml^2) - m^2 l^2 \cos^2\theta$, $a_1 = ml$, $a_2 = \dfrac{k}{M+m}$ and $a_3 = \dfrac{k\,ml}{(M+m)^2}$. The output of the system is $y = z_1$. Therefore, the zero dynamics of this system is

$$\begin{cases} \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = -a_2\eta_1. \end{cases}$$

Since $a_2 > 0$, the zero dynamics is unstable and the system is non-minimum phase. The linearised model of the TORA system is

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -\hat{a}_1^2 \hat{a}_2 \hat{\phi}^{-1}(0) z_1 + \hat{k}\hat{a}_1\hat{\phi}^{-1}(0)\eta_1 + (M+\hat{m})\hat{\phi}^{-1}(0)\, u \\ \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = -\hat{a}_2\eta_1 + \hat{a}_3 z_1, \end{cases}$$

where $\hat{m}$, $\hat{k}$, $\hat{J}$, $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$ and $\hat{\phi}$ are the estimates of the parameters m, k, J, a_1, a_2, a_3 and ϕ respectively. Hence, the matched and unmatched uncertainties can be represented as

$$\Delta(\boldsymbol{\eta}, \mathbf{z}, u) = \frac{1}{(\hat{M}+\hat{m})\hat{\phi}^{-1}(0)} \Big\{ \Big[(M+m)\phi^{-1}(z_1) - (\hat{M}+\hat{m})\hat{\phi}^{-1}(0) \Big] u + \Big[a_1\phi^{-1}(z_1)\cos z_1 - \hat{a}_1\hat{\phi}^{-1}(0) \Big]\eta_1$$
$$- m^2 l^2 \phi^{-1}(z_1) z_2^2 \sin z_1 \cos z_1 - \Big[a_1^2 a_2 \phi^{-1}(z_1)\sin z_1 \cos z_1 - \hat{a}_1^2\hat{a}_2\hat{\phi}^{-1}(0) z_1 \Big] \Big\}$$

$$\Delta_\eta(\boldsymbol{\eta}, z_1) = -(a_2 - \hat{a}_2)\eta_1 + a_3 \sin z_1 - \hat{a}_3 z_1.$$

Note that, Assumption 1 is satisfied; that is

$$\frac{\partial f(z,\eta,u)}{\partial u} = (M+m)\phi^{-1}(z_1) > 0$$

In addition, assume that the estimates of the model parameters are obtained such that the contractive mapping condition (14) is satisfied. That is,

$$(M+\hat{m})\hat{\phi}^{-1}(0) \geq 0.5(M+m)\phi^{-1}(z_1)$$

The sufficient conditions that ensure this assumption are $\hat{m} < 2m$ and $\hat{J} < 2J$, which are very mild conditions.

The NN is a MLP type and comprises of 10 neurons in one hidden layer with tangent hyperbolic as the activation functions. The weights are initialised randomly using small numbers. The input vector to the NN is

$$\xi = [1,\ y(t),\ y(t-T_d),\ y(t-2T_d),\ y(t-3T_d),\ u(t-T_d),\ u(t-2T_d)]^T,$$

with $T_d = 10$ m/sec. The learning coefficients are selected as

$$\gamma_w = \gamma_V = 3,\ \gamma_\varphi = 2,\ \Gamma_\lambda = \begin{bmatrix} 0.05 & 0 \\ 0 & 1 \end{bmatrix}$$

and $k_w = k_V = 1.2$. Moreover, the controller and observer gains are

$$\mathbf{k}_c = [-4.64, -1,\ -298.6, 6.96],\ \mathbf{k}_o = [32, 594.2,\ -2.14, 38.4]$$

For comparison, the simulations have been carried out using the same parameters and initial conditions as in [13]

$J = 0.0002175$ kg/m^2, $M = 1.3608$ kg, $m = 0.096$ kg, $l = 0.0592$ m, and $k = 186.3$ N/m,

$$\eta_1(0) = 0.025\ \text{m},\ \eta_2(0) = 0\ \text{m/sec.},\ z_1(0) = 0\ \text{rad},\ z_2(0) = 0\ \text{rad/sec}.$$

Simulation results are depicted in Figs. 4-8. Fig. 4 shows responses of the closed-loop system for x and θ using two different modes. First, only the proposed combined control law has been used without the unmatched uncertainty approximation (i.e. $y_d = 0$). Then, the proposed y_d has been employed. Note that when the unmatched uncertainty is compensated by y_d the responses converge faster. The closed-loop system responses are depicted in Figs. 5 and 6. Approximation of the matched uncertainty Δ using $u_{ad} + u_R$, the normalised norm of adaptive weights and state estimation errors are presented in Fig. 5. Fig. 6 shows the filtered reference signal y^* and the validity of the conic sector bound (5) on the unmatched uncertainty. Note that $c_1 = 0.2 \le 1$, which satisfies the required condition in Assumption 2.

Fig. 7 shows the response of the closed-loop system for x, θ, the control torque τ and the convergence rate of x and θ in the logarithmic scale. Note that the convergence rate of the proposed approach is faster than that of the backstepping-based controller using the method in [13].

To verify the robustness of the proposed controller, the simulation is repeated in the presence of parameters uncertainties with $\hat{m} = 1.1m$, $\hat{k} = 1.1k$, and $\hat{J} = 1.1J$. Note that these uncertainties can be embedded in the matched and unmatched uncertainties and since the proposed approach compensates these uncertainties adaptively, the approach is robust against parameters uncertainties. The simulation results, presented in Fig. 8, illustrate the robustness of the closed-loop system. As this figure shows, the proposed controller stabilises the system while the classical backstepping controller does not stabilise the system. In particular, the angular movement θ does not tend to the equilibrium point. In general, the performance of the backstepping controller for the class of the system in the form of (1) is inappropriate in comparison with the proposed NN controller.

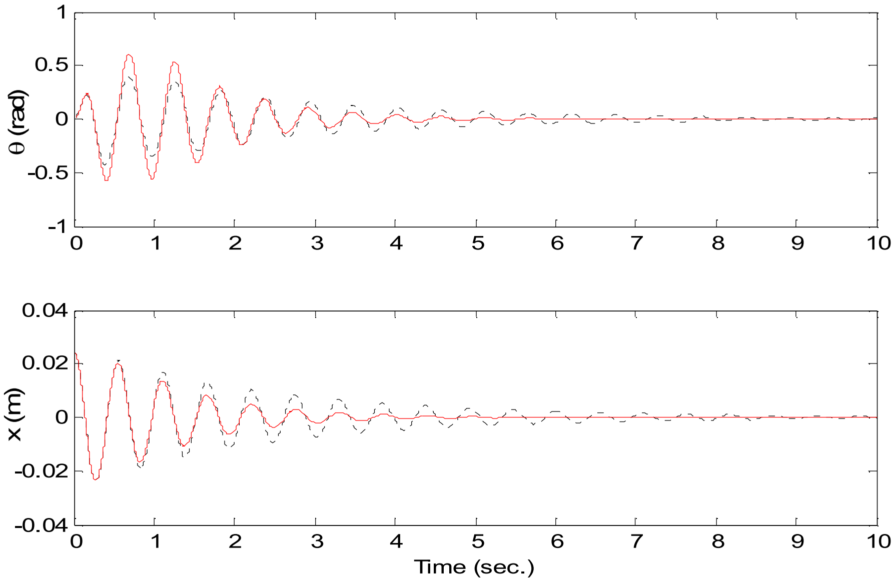

Fig. 4. Response of the TORA system in which dashed and solid lines present the system response without the unmatched uncertainty compensation (i.e. when $y_d = 0$) and with unmatched uncertainty cancellation using the proposed y_d , respectively.

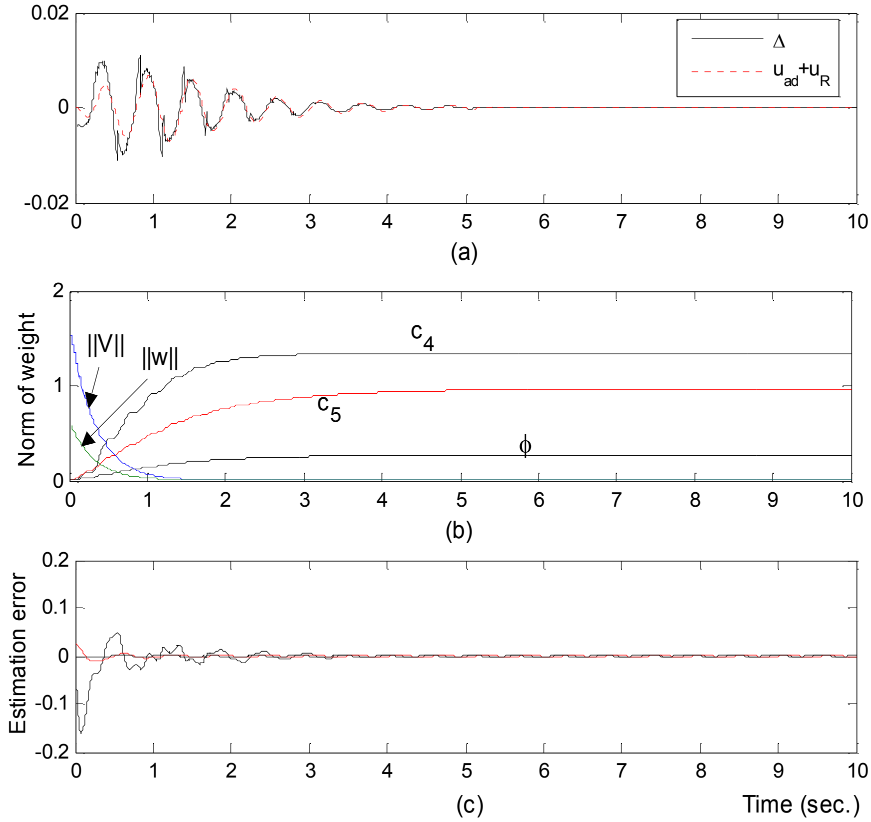

Fig. 5. The closed-loop signals of the TORA system: (a) Matched uncertainty cancellation; (b) Normalized norm of weights; (c) States estimation error

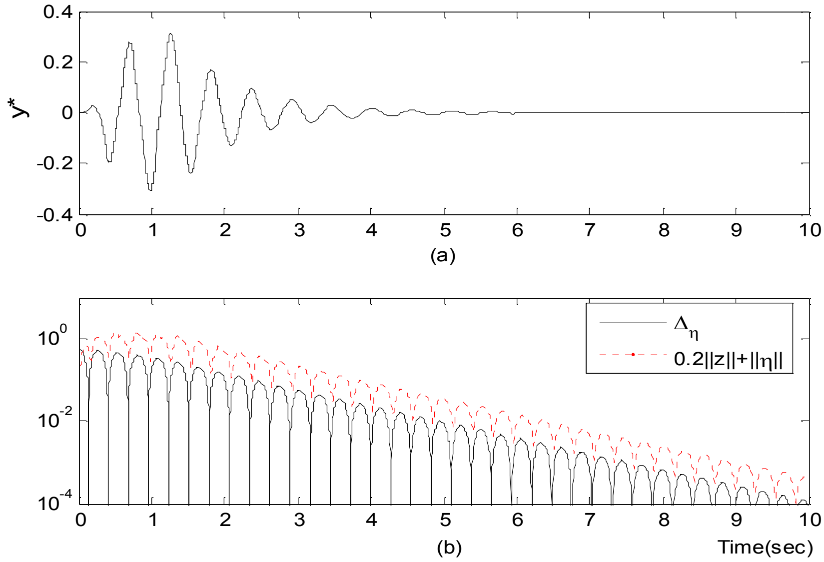

Fig. 6. The closed-loop signals of the TORA system: (a) the filtered reference signal, (b) the validity of the conic sector bound of the unmatched uncertainty

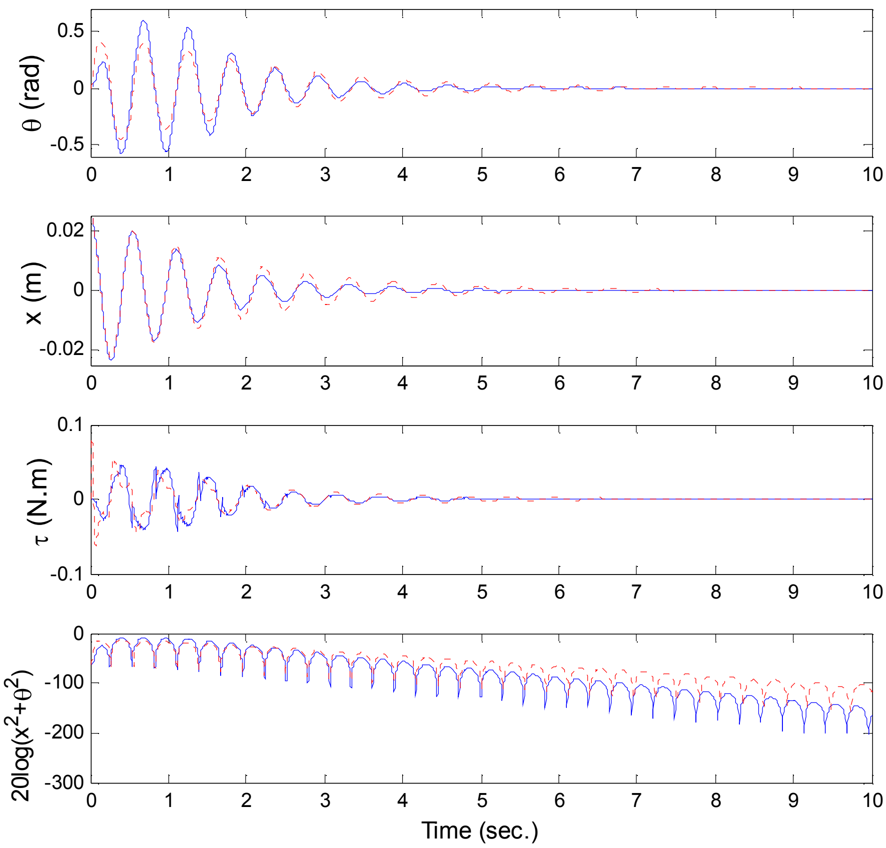

Fig. 7. The responses of the TORA without the parameter uncertainties in which the solid and dotted lines show the action of the proposed and backstepping controllers, respectively.

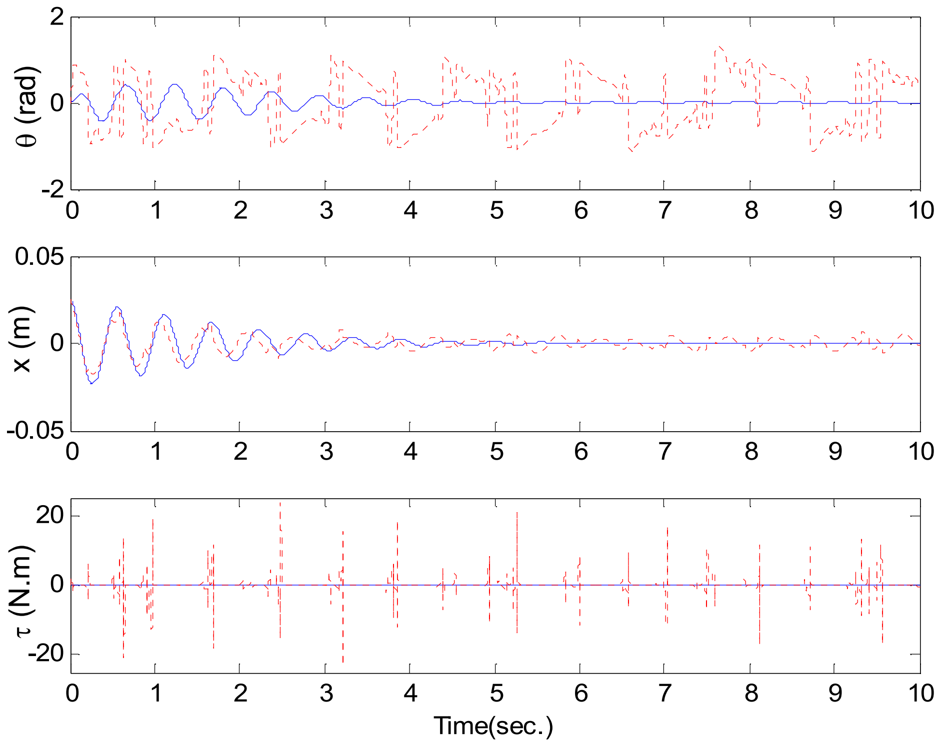

Fig. 8. The responses of the TORA with parameters uncertainties in which solid and dotted lines show the action of the proposed and the classical backstepping controllers, respectively.

6 Conclusions

A direct adaptive output feedback stabilization method using neural networks techniques for non-minimum phase nonlinear systems has been proposed in this paper. The proposed method is robust with respect to the matched and unmatched uncertainties, and relies on the state estimation. The approach can be applied to a general class of uncertain nonlinear systems, from which a linear approximation can be derived. Using the Lyapunov direct method the ultimate boundedness of the states and the NN weights have been achieved. The theoretical results have been successfully applied to a TORA model, which show that the method yields the desired responses in comparison with other nonlinear control design methods such as the backstepping method.

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