Neuro-Adaptive Output Feedback Control for a Class of Nonlinear Non-minimum Phase Systems

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Abstract This paper presents an adaptive output-feedback control method for non-affine nonlinear non-minimum phase systems that have partially known Lipschitz continuous functions in their arguments. The proposed controller is comprised of a linear, a neuro-adaptive and an adaptive robustifying control term. The adaptation law for the neural network weights is obtained using the Lyapunov's direct method. One of the main advantageous of the proposed method is that the control law does not depend on the state estimation. This task is accomplished by introducing a strictly positive-real augmented error dynamic and using the Leftshetz-Kalman-Yakobuvich lemma. The ultimate boundedness of the error signals will be shown analytically using the extension of Lyapunov theory. The effectiveness of the proposed scheme will be shown in simulations for the benchmark problem Translational Oscillator/Rotational Actuator (TORA) system.

Key words: neural networks , nonlinear non-minimum system, adaptive control, output feedback, strictly positive real.

1 Introduction

Control of nonlinear non-minimum phase systems is a difficult task in control theory. This problem has been an active research area for many years. Several fundamental methods have been proposed in this area based on the state-feedback control, including output redefinition and zero assignment [1-3], stable inversion, and iterative learning control for systems with predefined reference signals [4-6]. Moreover, sliding mode control method [7], neural networks, and fuzzy logic [8-10] have been applied successfully to control uncertain non-minimum phase systems.

In the case of output feedback control, the problem is more complicated. Unlike linear systems, state observation for nonlinear systems is often not an easy task, even for some simple nonlinear systems. For some nonlinear systems, the observer error dynamic becomes linear [11]. The main issue in output feedback control of non-minimum phase systems stems from the fact that information on the state variables, associated with the zero dynamics, is vital in the control design. Marino and Tomei have considered a special case of this format for the systems in output feedback form [12].

Recently, some researchers have proposed methods for output-feedback stabilization for uncertain non-minimum phase systems. Isidori [13] has proposed a solution for semi-global output-feedback stabilization of non-minimum phase systems based on auxiliary constructions using a high-gain observer. Global output-feedback stabilization using the backstepping and the small-gain techniques are employed by Karagiannis et al. [14], and Wang et al. [15]. Ding has proposed a design method for semi-global stabilization of a class of non-minimum phase nonlinear systems that can be transformed to the global normal form as well as to the form of linear observer error dynamic [16]. Sliding mode observer and output feedback sliding mode control is applied by Yang [17]. Hovakimian et al. have considered the output feedback control of a more general class of non-minimum phase nonlinear system based on universal approximation properties of NN and using a high-order error observer [18].

This paper presents an adaptive output-feedback control method for observable and stabilizable nonlinear non-minimum phase systems. The proposed method does not relay on state estimation. Moreover, only an approximate linear model of the nonlinear system is required in the design procedure with some mild conditions. This linear model should present the non-minimum phase zeros of the nonlinear system with some acceptable accuracy. In fact, there is a conic sector bound on the modelling error of the non-minimum phase zeros that is referred to as the unmatched uncertainty. Hence, the proposed approach can be applied to uncertain systems, which have partially known Lipschitz continuous functions in their arguments.

In the design procedure, first, a linear tracking controller is designed for the linear approximation of the system such that the closed-loop system represents a desired reference model. Then, the linear controller is augmented with a neuro-adaptive element, which is used to approximate the matched uncertainty. The NN operates over tapped-delay units, comprised of the system input/output signals. The adaptation rules for the NN weights usually require the state variables of the system [19, 20]. However, designing an observer to estimate the state variables is not an easy task for nonlinear systems. To avoid state estimation, this paper introduces a Strictly Positive Real (SPR) augmented error dynamic and uses the Leftshetz-Kalman-Yakobuvich (LKY) Lemma to substitute the state variables with this augmented error, which comprises of two parts: the available part and the unavailable part. The latter part increases the ultimate bound of the error and decrease the rate of convergence of the error signals. To compensate the unavailable part of the augmented error as well as the approximation error of the NN, an additional adaptive robustifying control term u_R will be incorporated into the control law.

This paper is organized as follows: Section 2 describes the class of nonlinear systems to be controlled and defines the problem of tracking. Section 3 presents procedure for the controller design and approximation properties of the NN. Section 4 provides the analytical work about stability of the closed-loop system. Section 5 gives simulation example, which illustrates the effectiveness of the proposed controller. Finally, Section 6 concludes the paper.

2 Problem Statement

Consider the nonlinear SISO system in the following normal form [12]:

$$\begin{cases} \dot{z}_i = z_{i+1} \quad 1 \le i \le r-1 \\ \dot{z}_r = f(\mathbf{z}, \boldsymbol{\eta}, u) \\ \dot{\boldsymbol{\eta}} = \mathbf{v}(\mathbf{z}, \boldsymbol{\eta}) \\ y = z_1, \end{cases} \tag{1}$$

where r is the relative degree, $\boldsymbol{\eta} \in \Omega_\eta \subset R^{n-r}$ is the state vector associated with the internal dynamics, $\mathbf{z} = [z_1, \cdots, z_r]^T \in \Omega_z \subset R^r$, Ω_η and Ω_z are the compact sets of the operating regions, and $u \in R$ and $y \in R$ are the input and the output of the system, respectively. The mappings $f : R^{n+1} \to R$ and $\mathbf{v} : R^n \to R^{n-r}$ are partially known Lipschitz continuous functions of their arguments with the initial conditions $f(\mathbf{0}, \mathbf{0}, 0) = 0$ and $\mathbf{v}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. Note that the system in (1) may be non-minimum phase. Hence, the stability assumption on the zero dynamics is not required. The goal is to design a combined adaptive controller such that the output tracks a desired reference signal y_d with bounded tracking error. Moreover, the internal dynamics of system must remain stable. Next section presents various features of the proposed control method.

3 Controller Design

3.1 Model expansion

Since the mappings $f : R^{n+1} \to R$ and $\mathbf{v} : R^n \to R^{n-r}$ are partially known Lipschitz continuous functions of their arguments, system in (1) can be represented as the following expanded model:

$$\begin{cases} \dot{z}_i = z_{i+1} \quad 1 \le i \le r-1 \\ \dot{z}_r = \mathbf{m}^T\mathbf{z} + \mathbf{n}^T\boldsymbol{\eta} + b\psi(\mathbf{z}, \boldsymbol{\eta}, u) \\ \dot{\boldsymbol{\eta}} = \mathbf{F}\boldsymbol{\eta} + \mathbf{G}\mathbf{z} + \boldsymbol{\Delta}_{\boldsymbol{\eta}}(\mathbf{z}, \boldsymbol{\eta}) \\ y = z_1, \end{cases} \tag{2}$$

where **m** and **n** are coefficient vectors, **F** and **G** are matrices, all with appropriate dimensions, $\psi(\mathbf{z},\boldsymbol{\eta},u)$ is an uncertain function such that $\partial\psi/\partial u \neq 0$ over the compact set $\Omega_z \times \Omega_\eta \times R$ and with a known upper bound for $\partial\psi/\partial u$. Also $\boldsymbol{\Delta}_{\boldsymbol{\eta}}(\mathbf{z},\boldsymbol{\eta})$ denotes the vector of internal-dynamic modelling error or the unmatched uncertainty, which is assumed to be bounded with a conic sector bound as in the following assumption.

ASSUMPTION 1: The unmatched uncertainty is bounded with a sector bound as

$$\left\|\boldsymbol{\Delta}_{\boldsymbol{\eta}}(\mathbf{z},\boldsymbol{\eta})\right\| \leq d_0 + d_1\left\|\mathbf{z}\right\| + d_2\left\|\boldsymbol{\eta}\right\| \qquad \forall(\mathbf{z},\boldsymbol{\eta}) \in \Omega_z \times \Omega_\eta , \tag{3}$$

where d_i $(i = 0,1,2)$ are known positive constants.

Let $\hat{\psi}(y,u)$ be the best available approximation of $\psi(\mathbf{z},\boldsymbol{\eta},u)$. Consider the following pseudo control:

$$\upsilon = \hat{\psi}(y,u). \tag{4}$$

It should be mentioned that the model approximation function $\hat{\psi}(\cdot,\cdot)$ should be invertible with respect to *u*, allowing the actual control input to be computed by

$$u = \hat{\psi}^{-1}(y,\upsilon) \tag{5}$$

Then, the modelling error of ψ is

$$\Delta(\mathbf{z},\boldsymbol{\eta},u) = \psi(\mathbf{z},\boldsymbol{\eta},u) - \hat{\psi}(y,u). \tag{6}$$

Let define the pseudo control law in (4) as

$$\upsilon := u_L - u_{ad} - u_R , \tag{7}$$

where u_L, u_{ad} and u_R are the linear, the neuro-adaptive and the robustifying control terms, respectively. Using (4), (6) and (7), the system in (2) can be described as

$$\begin{cases} \dot{z}_i = z_{i+1} & 1 \leq i \leq r-1 \\ \dot{z}_r = \mathbf{m}^T\mathbf{z} + \mathbf{n}^T\boldsymbol{\eta} + bu_L + b\big(\Delta(\mathbf{z},\boldsymbol{\eta},u) - u_{ad} - u_R\big) & \\ \dot{\boldsymbol{\eta}} = \mathbf{F}\,\boldsymbol{\eta} + \mathbf{G}\,\mathbf{z} + \boldsymbol{\Delta}_{\boldsymbol{\eta}}(\mathbf{z},\boldsymbol{\eta}) & \\ y = z_1, & \end{cases} \tag{8}$$

Here, the modelling error Δ acts as the matched uncertainty. Notice that, since the system is non-minimum phase, some eigenvalues of $\mathbf{F}$ have positive real parts.

3.2 Construction of error dynamic

Consider the linear model of the system in (8) as

$$\begin{cases} \dot{z}_i^l = z_{i+1}^l & 1 \le i \le r-1 \\ \dot{z}_r^l = \mathbf{m}^T \mathbf{z}_l + \mathbf{n}^T \boldsymbol{\eta}_l + b u_L \\ \dot{\boldsymbol{\eta}}_l = \mathbf{F}\,\boldsymbol{\eta}_l + \mathbf{G}\,\mathbf{z}_l \\ y_l = z_1. \end{cases} \tag{9}$$

ASSUMPTION 2: There exists an output feedback tracking linear controller for the linear dynamic (9) that satisfies the performance requirements.

A necessary condition for existence of such controller is that the system has no zeros at the origin. Let this linear controller be defined as [21]

$$u_L \triangleq -\frac{S(s)}{R(s)} y + \frac{T(s)}{R(s)} y_d \,, \tag{10}$$

where y_d is the desired output, S and R are designed to assure stability of the close-loop system, and T is designed to achieve the desired tracking. The state-space form of this controller is

$$\begin{cases} \dot{\mathbf{x}}_c = \mathbf{A}_c \mathbf{x}_c + \mathbf{b}_{c1}\, y + \mathbf{b}_{c2}\, y_d \\ u_L = \mathbf{c}_c \mathbf{x}_c + d_{c1}\, y + d_{c2}\, y_d. \end{cases} \tag{11}$$

The linear model (9), regulated by (11), defines a closed-loop reference model as

$$\begin{cases} \dot{\mathbf{x}}_l = \mathbf{A}_{cl} \mathbf{x}_l + \mathbf{b}_d\, y_d \\ y_l = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \mathbf{x}_l, \end{cases} \tag{12}$$

where $\mathbf{x}_l^T \triangleq [\mathbf{z}_l^T, \boldsymbol{\eta}_l^T, \mathbf{x}_{cl}^T]$, in which $\mathbf{x}_{cl}$ denotes the vector of state variables of the controller (11), when applied to linear system (9).

The closed-loop dynamic of the nonlinear system in (8), when regulated by the linear controller (11), can be written as

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_{cl}\mathbf{x} + \mathbf{b}_d y_d + \mathbf{b}_{cl}\left(u_{ad} + u_R - \Delta\right) - \mathbf{G}_{cl}\boldsymbol{\Delta}_\eta \\ y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}\mathbf{x}, \end{cases} \tag{13}$$

where $\mathbf{x}^T \triangleq [\mathbf{z}^T, \boldsymbol{\eta}^T, \mathbf{x}_c^T]$,

$$\mathbf{A}_{cl} = \begin{bmatrix} \mathbf{M} + \mathbf{b}d_{c1}\mathbf{c} & \mathbf{N} & \mathbf{b}\mathbf{c}_c \\ \mathbf{G} & \mathbf{F} & 0 \\ \mathbf{b}_{c1}\mathbf{c} & 0 & \mathbf{A}_c \end{bmatrix}, \quad \mathbf{G}_{cl} = \begin{bmatrix} \mathbf{0}_{r\times(n-r)} \\ -\mathbf{I}_{(n-r)\times(n-r)} \\ \mathbf{0}_{n_c\times(n-r)} \end{bmatrix},$$

$$\mathbf{b}_{cl} = \begin{bmatrix} -\mathbf{b}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathrm{T}}, \qquad \mathbf{b}_d = \begin{bmatrix} \mathbf{b}^T d_{c2} & \mathbf{0} & \mathbf{b}_{c2}^T \end{bmatrix}^T,$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{0}_{(r-1)\times 1} & \mathbf{I}_{(r-1)\times(r-1)} \\ \hline \multicolumn{2}{c}{\mathbf{m}^T} \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} \mathbf{0}_{(r-1)\times(n-r)} \\ \mathbf{n}^T \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \mathbf{0}_{1\times(r-1)} & b \end{bmatrix}^{\mathrm{T}}, \qquad \mathbf{c} = \begin{bmatrix} 1 & \mathbf{0}_{1\times(r-1)} \end{bmatrix}.$$

Using (12) and (13), and defining the error vector as

$$\boldsymbol{\xi} \triangleq \mathbf{x}_l - \mathbf{x}, \tag{14}$$

the error dynamic can be represented as

$$\begin{cases} \dot{\boldsymbol{\xi}} = \mathbf{A}_{cl}\boldsymbol{\xi} + \mathbf{b}_{cl}\left(\Delta - u_{ad} - u_R\right) + \mathbf{G}_{cl}\boldsymbol{\Delta}_\eta \\ \rho = \mathbf{c}_{cl}\boldsymbol{\xi}, \end{cases} \tag{15}$$

where ρ is the available output error signal containing the output tracking error $e = y_l - y$ and the state variables associated with the controller error $\mathbf{x}_{cl} - \mathbf{x}_c$. Hence, the output vector has the following form:

$$\mathbf{c}_{cl} = \begin{bmatrix} c_1 & \mathbf{0}_{1\times(n-1)} & c_{n+1} & \cdots & c_{n+n_c} \end{bmatrix}, \tag{16}$$

where n_c is the order of the dynamic tracking controller in (11). Note that if ξ_i $(2 \le i \le n)$ is also available (i.e., beside the system output, other state variables of the system are also accessible), then the corresponding c_i should be set to a suitable non-zero value, which will be determined in next section.

The objective of this paper is to design u_{ad} and u_R such that the system output y tracks the reference signal y_d. The error dynamic $\boldsymbol{\xi}$ is derived in this section and in Section 4, the ultimate boundedness of this error signal will be shown using the

Lyapunov's direct method. Then, since y_l tracks y_d by Assumption 2, this ensures that y tracks y_d with a bounded error trajectory.

3.3. CONSTRUCTION OF SPR TRANSFER FUNCTION

Consider the error dynamic in (15). If the following transfer function is Strictly Positive Real (SPR):

$$G(s) = \frac{\rho}{\left(\Delta - u_{ad} - u_R\right)}, \tag{17}$$

then, by applying the Leftshetz-Kalman-Yakobuvich (LKY) lemma, the measurable signal ρ can be used instead of the state variable vector $\boldsymbol{\xi}$ to construct the control terms u_{ad} and u_R. However, because of the non-minimum phase properties of the system in (1), the closed-loop transfer function (17) cannot be SPR [22].

Now, let define the augmented error dynamic output as

$$\rho_{ag} = \mathbf{c}_{ag}\boldsymbol{\xi} = \rho + \rho_a \tag{18}$$

such that $G_{ag}(s) = \mathbf{c}_{ag}\left(s\mathbf{I} - \mathbf{A}_{cl}\right)^{-1}\mathbf{b}_{cl}$ represents a SPR transfer function. Since $\mathbf{A}_{cl}$ is a stable matrix, one can always find a $\mathbf{c}_{ag}$ to ensure the strictly positive realness of $G_{ag}(s)$. The error dynamic output in (18) can be considered as a combination of the available signal ρ defined in (15) and an unavailable error signal $\rho_a = \mathbf{c}_a\boldsymbol{\xi}$, which can be considered as a disturbance term.

It will be shown in Section 4 that the ultimate error bound increases and the rate of the error convergence decreases proportional to $|\rho_a|$. Let define the following bound for $|\rho_a|$:

$$|\rho_a| \le \lambda_0 + \lambda_1 |\rho|, \tag{19}$$

where λ_0 is a positive constant and $0 \le \lambda_1 < 1$. As it will be shown in the proof of the theorem, the ultimate error bound increases and the rate of error convergence decreases proportional to λ_0 while $\lambda_1|\rho|$ can be compensated using the robustifying control term u_R. Note that, it is always possible to found such constants; and in the worst case, $\lambda_0 = \sup_t (|\rho_a(t)|)$ with $\lambda_1 = 0$ satisfies inequality (19). Hence, to achieve the smallest value of λ_0, it is desired to select output vectors such that $|\rho_a|$ (or alternatively $\|\mathbf{c}_a\|$) is minimized with respect to $\|\mathbf{c}_{cl}\|$.

$G_{ag}(s)$ is SPR if and only if it complies with the LKY lemma. I.e., there should exists a matrix $\mathbf{Q} = \mathbf{Q}^T > 0$ such that the solution $\mathbf{P}$ of

$$\mathbf{A}_{cl}^T \mathbf{P} + \mathbf{P}\mathbf{A}_{cl} = -\mathbf{Q} \tag{20}$$

is positive definite and

$$\mathbf{c}_{ag}^T = \mathbf{P}\mathbf{b}_{cl}. \tag{21}$$

For practical purposes, a simple optimization algorithm is given in Appendix to find the appropriate values for $\mathbf{c}_{cl}$ and $\mathbf{c}_a$.

3.4 Neural network-based adaptive controller design

The term u_{ad} in the control law (8), is designed to approximate the modelling error $\Delta(\mathbf{z}, \boldsymbol{\eta}, u)$. Hence, there exists a fixed-point problem as

$$u_{ad}(t) = \Delta\left(\mathbf{z}, \boldsymbol{\eta}, \hat{\psi}^{-1}(y, -u_{ad}(t) - u_R + u_L)\right) \tag{22}$$

The following assumption provides conditions that guarantee the existence and uniqueness of a solution for u_{ad} [23].

ASSUMPTION 3**:** The map $u_{ad} \rightarrow \Delta$ is a contraction over the entire input domain.

This means the following inequality should be satisfied

$$\left|\frac{\partial \Delta}{\partial u_{ad}}\right| < 1. \tag{23}$$

Substituting (4), (6) and (7) into (23), yields

$$\left|\frac{\partial \Delta}{\partial u_{ad}}\right| = \left|\frac{\partial\left(\psi(\mathbf{z}, \boldsymbol{\eta}, u) - \hat{\psi}(y, u)\right)}{\partial u}\frac{\partial u}{\partial \upsilon}\frac{\partial \upsilon}{\partial u_{ad}}\right| = \left|-\frac{\partial(\psi - \hat{\psi})}{\partial u}\frac{\partial u}{\partial \hat{\psi}}\right| < 1 \tag{24}$$

It is straightforward to show that condition (24) is equivalent to the following three conditions:

$$\operatorname{sgn}\left(\frac{\partial \psi}{\partial u}\right)=\operatorname{sgn}\left(\frac{\partial \hat{\psi}}{\partial u}\right),$$
$$\left|\frac{\partial \hat{\psi}}{\partial u}\right|>0.5\left|\frac{\partial \psi}{\partial u}\right| \tag{25}$$

In the following lemma, it will be shown that a neural network based on the input-output data only can approximate the modelling error $\Delta(\mathbf{z},\boldsymbol{\eta},u)$. Moreover, it will be proved that if any non-affine system satisfies conditions (23), then it is unnecessary to use $u_{ad}(t)$ as an input signal to NN.

Lemma 1 *If conditions (23) are satisfied, then, the modelling error* $\Delta(\mathbf{z},\boldsymbol{\eta},u)$ *can be approximated by a single hidden layer MultiLayer Perceptron (MLP) as* $\mathbf{w}^{*T}\boldsymbol{\sigma}\left(\mathbf{V}^{*T}\boldsymbol{\zeta}\right)$, *where* $\mathbf{w}^{*}\in R^{m}$ *is the vector containing synaptic weights of the output layer,* $\mathbf{V}^{*}\in R^{N\times m}$ *is the matrix containing the weights of the hidden layer,* $\boldsymbol{\sigma}=\left[\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_m\right]^T$ *is the vector function containing the nonlinear functions in the neurons of the hidden layer, and* $\boldsymbol{\zeta}\in R^{N}$ *is the input vector, which is equal to* $\boldsymbol{\zeta}=\left[1\quad \bar{\mathbf{y}}\quad \bar{\mathbf{u}}_{\alpha}\quad \bar{\mathbf{u}}_{ad}\right]^T$, *where*

$$\begin{aligned}\bar{\mathbf{y}}&=\left[y(t)\quad\cdots\quad y(t-T_d(n_1-1))\right]\\ \bar{\mathbf{u}}_{\alpha}&=\left[u_{\alpha}(t)\quad\cdots\quad u_{\alpha}(t-T_d(n_1-r-1))\right]\\ \bar{\mathbf{u}}_{ad}&=\left[u_{ad}(t-T_d)\quad\cdots\quad u_{ad}(t-T_d(n_1-r-1))\right]\end{aligned} \tag{26}$$

in which $u_{\alpha}=\upsilon+u_{ad}=u_L-u_R$.

Proof Under the observability condition of the system in (1), it has been shown by Lavertsky et al. that the continuous-time dynamic $\Delta(\mathbf{z},\boldsymbol{\eta},u)$ can be approximately reconstructed using delayed inputs-outputs as [24]:

$$\Delta(\mathbf{z},\boldsymbol{\eta},u)=\mathrm{F}\left(\bar{y},\bar{\upsilon}\right)+\varepsilon_1, \tag{27}$$

where

$\bar{\upsilon}=\left[\upsilon(t)\quad \upsilon(t-T_d)\quad\cdots\quad \upsilon(t-T_d(n_1-r-1))\right]$, $n_1\geq n$, and $|\varepsilon_1|\leq\varepsilon_{1M}$, in which ε_{1M} is proportional to the sampling time interval T_d, and $\mathrm{F}(\cdot,\cdot)$ is a non-linear function of its arguments. Hence, the approximation error ε_1 can be ignored by selecting T_d sufficiently small. On the other hand, conditions (23) guarantee existence and uniqueness of a solution for u_{ad} from the following equation:

$$\Phi(\mathbf{z},\boldsymbol{\eta},u,u_{ad}) \triangleq \Delta(\mathbf{z},\boldsymbol{\eta},u) - u_{ad}(t) = 0. \tag{28}$$

Differentiating Φ with respect to u_{ad} and using (4) and (6) yields

$$\begin{aligned}\frac{\partial}{\partial u_{ad}}\Phi(\mathbf{z},\boldsymbol{\eta},u,u_{ad}) &= \frac{\partial}{\partial u_{ad}}\left(\Delta(\mathbf{z},\boldsymbol{\eta},u) - u_{ad}\right) = \frac{\partial}{\partial u_{ad}}\left(\psi(\mathbf{z},\boldsymbol{\eta},u) - \upsilon\right) - 1 \\ &= \frac{\partial}{\partial \upsilon}\left(\psi(\mathbf{z},\boldsymbol{\eta},u) - \upsilon\right)\frac{\partial \upsilon}{\partial u_{ad}} - 1 = \left(\frac{\partial}{\partial \upsilon}\psi(\mathbf{z},\boldsymbol{\eta},u) - 1\right)(-1) - 1 \\ &= -\left(\frac{\partial}{\partial u}\psi(\mathbf{z},\boldsymbol{\eta},u)\right)\left(\frac{\partial u}{\partial \upsilon}\right),\end{aligned} \tag{29}$$

which is nonzero over the set $\Omega_z \times \Omega_\eta \times R$ according to (4) and definition of ψ given in (2). Thus, from (29), using (27), and according to the implicit function theorem, the following equation:

$$\Delta(\mathbf{z},\boldsymbol{\eta},u) - u_{ad}(t) = \mathrm{F}\left(\bar{y},\bar{u}_\alpha + \bar{u}_{ad} + u_{ad}(t)\right) - u_{ad}(t) = 0, \tag{30}$$

implies that there exists a unique solution for u_{ad} over the set $\Omega_z \times \Omega_\eta \times R$ as

$$u_{ad}(t) = \Gamma(\bar{y},\bar{u}_\alpha,\bar{u}_{ad}). \tag{31}$$

Now, substituting (31) into (30) yields

$$\Delta(\mathbf{z},\boldsymbol{\eta},u) = \Gamma(\zeta). \tag{32}$$

On the other hand, any sufficiently smooth function can be approximated on a compact set with arbitrarily bounded error by a suitable large MLP. Therefore, a set of ideal weights $\mathbf{w}^*$ and $\mathbf{V}^*$ on the compact set Ω_Δ exist such that

$$\Delta(\mathbf{z},\boldsymbol{\eta},u) = \mathbf{w}^{*T}\boldsymbol{\sigma}(\mathbf{V}^{*T}\zeta) + \varepsilon_2, \quad \forall\left(\mathbf{z},\boldsymbol{\eta},u\right) \in \Omega_\Delta, \tag{33}$$

where $|\varepsilon_2| \le \varepsilon_{2M}$ in which ε_{2M} depends on the network architecture. The ideal constant weights $\mathbf{w}^*$ and $\mathbf{V}^*$ are defined as

$$\left(\mathbf{w}^*,\mathbf{V}^*\right) \triangleq \arg\min_{(\mathbf{w},\mathbf{V})\in\Omega_\mathbf{w}}\left\{\sup_{\zeta\in\Omega_\zeta}\left|\mathbf{w}^T\boldsymbol{\sigma}\left(\mathbf{V}^T\zeta\right) - \Gamma(\zeta)\right|\right\}, \tag{34}$$

where $\Omega_\mathbf{w} = \left\{(\mathbf{w},\mathbf{V}) \,\middle|\, \|\mathbf{w}\| \le M_\mathbf{w}, \|\mathbf{V}\|_\mathrm{F} \le M_\mathbf{V}\right\}$, in which $M_\mathbf{w}$ and $M_\mathbf{V}$ are positive numbers and $\|\cdot\|_\mathrm{F}$ denotes the Frobenius norm. □

Since Δ can be modelled using a MLP, the adaptive control term is proposed as

$$u_{ad} = \mathbf{w}^T \boldsymbol{\sigma}\left(\mathbf{V}^T \zeta\right), \tag{35}$$

where $\mathbf{w}$ and $\mathbf{V}$ are the actual weights of their corresponding ideal weights defined in (34). Hence, in practice, the weights of the NN may be different from the ideal ones. Therefore, an approximation error exists.

Lemma 2 If the activation function $\sigma(\cdot)$ is of sigmoid type functions, then the approximation error arising from the difference between (33) and (35) satisfies the following equation:

$$\Delta(\mathbf{z}, \boldsymbol{\eta}, u) - u_{ad} = \tilde{\mathbf{w}}^T \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathrm{v}} \mathbf{V}^T \zeta\right) + \mathrm{tr}\left(\tilde{\mathbf{V}}^T \zeta \mathbf{w}^T \boldsymbol{\sigma}_{\mathrm{v}}\right) + \delta(t), \tag{36}$$

where

$$\begin{aligned} &|\delta(t)| \le \left(\varepsilon_{2M} + 2\sqrt{m} M_{\mathrm{w}}\right) + M_{\mathrm{w}} \left\|\tilde{\mathbf{V}}\right\|_{\mathrm{F}} \|\zeta\| + M_{\mathrm{V}} \|\tilde{\mathbf{w}}\| \|\zeta\| \\ &\tilde{\mathbf{w}} = \mathbf{w}^* - \mathbf{w} \\ &\tilde{\mathbf{V}} = \mathbf{V}^* - \mathbf{V}, \end{aligned} \tag{37}$$

and $\boldsymbol{\sigma}_{\mathrm{v}} \triangleq \mathrm{diag}\left[\partial\sigma_1(v_1)/v_1 \quad \cdots \quad \partial\sigma_m(v_m)/v_m\right]$ is the derivative of $\boldsymbol{\sigma}$ with respect to the input signals v_i $(i = 1, \ldots, m)$, in which $[v_1 \cdots v_m]^T = \mathbf{V}^T \zeta$, and m denotes the number of neurons in the hidden layer.

Proof The Taylor series expansion of $\boldsymbol{\sigma}(\mathbf{V}^{*T} \zeta)$ gives

$$\boldsymbol{\sigma}(\mathbf{V}^{*T} \zeta) = \boldsymbol{\sigma}(\mathbf{V}^T \zeta + \tilde{\mathbf{V}}^T \zeta) = \boldsymbol{\sigma}(\mathbf{V}^T \zeta) + \boldsymbol{\sigma}_{\mathrm{v}}(\mathbf{V}^T \zeta) \tilde{\mathbf{V}}^T \zeta + \mathbf{O}(\cdot), \tag{38}$$

where $\mathbf{O}(\cdot) \in R^m$ denotes the vector associated with high order terms.

The activation function in the neurons of MLP is of sigmoid type functions (e.g. logistic function $\frac{1}{1 + e^{-\alpha v_i}}$ or $\tanh(\alpha v_i)$). Hence, $|\sigma_i| \le 1$ and $|\partial\sigma_i(v_i)/v_i| \le \alpha$. Consequently, it is straightforward to show that $\|\boldsymbol{\sigma}\| \le \sqrt{m}$ and $\|\boldsymbol{\sigma}_{\mathrm{v}}\| \le \alpha$. Using these inequalities and (38) it is easy to conclude that $\|\mathbf{O}(\cdot)\|$ is also bounded and can be represented as

$$\|\mathbf{O}(\cdot)\| = \left\|\boldsymbol{\sigma}(\mathbf{V}^{*T} \zeta) - \boldsymbol{\sigma}(\mathbf{V}^T \zeta) - \boldsymbol{\sigma}_{\mathrm{v}}(\mathbf{V}^T \zeta) \tilde{\mathbf{V}}^T \zeta\right\| \le 2\sqrt{m} + \alpha \left\|\tilde{\mathbf{V}}\right\|_{\mathrm{F}} \|\zeta\|. \tag{39}$$

Therefore, the approximation error can be calculated as

$$
\begin{aligned}
\Delta - u_{ad} &= \mathbf{w}^{*T}\boldsymbol{\sigma}(\mathbf{V}^{*T}\zeta) + \varepsilon_2 - \mathbf{w}^T\boldsymbol{\sigma}(\mathbf{V}^T\zeta) \\
&= \left(\mathbf{w}^T + \tilde{\mathbf{w}}^T\right)\left(\boldsymbol{\sigma}(\mathbf{V}^T\zeta) + \boldsymbol{\sigma}_{\mathrm{v}}(\mathbf{V}^T\zeta)\left(\mathbf{V}^{*T} - \mathbf{V}^T\right)\zeta + \mathbf{O}(\cdot)\right) + \varepsilon_2 - \mathbf{w}^T\boldsymbol{\sigma}(\mathbf{V}^T\zeta) \\
&= \tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathrm{v}}\mathbf{V}^T\zeta\right) + \mathbf{w}^T\boldsymbol{\sigma}_{\mathrm{v}}\tilde{\mathbf{V}}^T\zeta + \tilde{\mathbf{w}}^T\boldsymbol{\sigma}_{\mathrm{v}}\mathbf{V}^{*T}\zeta + \mathbf{w}^{*T}\mathbf{O}(\cdot) + \varepsilon_2 \\
&= \tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathrm{v}}\mathbf{V}^T\zeta\right) + \mathrm{tr}\left(\tilde{\mathbf{V}}^T\zeta\mathbf{w}^T\boldsymbol{\sigma}_{\mathrm{v}}\right) + \delta,
\end{aligned}
$$

where

$$
\delta \triangleq \tilde{\mathbf{w}}^T\boldsymbol{\sigma}_{\mathrm{v}}\mathbf{V}^{*T}\zeta + \mathbf{w}^{*T}\mathbf{O}(\cdot) + \varepsilon_2 .
$$

Now, using (34), (39), and the fact that $\|\boldsymbol{\sigma}_{\mathrm{v}}\| \le \alpha$, it gives

$$
\begin{aligned}
|\delta| &\le \|\tilde{\mathbf{w}}\|\|\boldsymbol{\sigma}_{\mathrm{v}}\|\|\mathbf{V}^*\|_{\mathrm{F}}\|\zeta\| + \|\mathbf{w}^*\|\|\mathbf{O}\| + \varepsilon_{2M} \\
&\le \alpha M_{\mathrm{V}}\|\tilde{\mathbf{w}}\|\|\zeta\| + M_{\mathrm{w}}\left(2\sqrt{m} + \alpha\|\tilde{\mathbf{V}}\|_{\mathrm{F}}\|\zeta\|\right) + \varepsilon_{2M} \\
&= \left(\varepsilon_{2M} + 2\sqrt{m}M_{\mathrm{w}}\right) + \alpha M_{\mathrm{w}}\|\tilde{\mathbf{V}}\|_{\mathrm{F}}\|\zeta\| + \alpha M_{\mathrm{V}}\|\tilde{\mathbf{w}}\|\|\zeta\|.
\end{aligned}
\qquad \square
$$

The adaptation rules for the weights of the neuro-adaptive controller u_{ad} in (35) is proposed as

$$
\begin{aligned}
\dot{\mathbf{w}} &= \gamma_{\mathrm{w}}\left(\rho\left(\boldsymbol{\sigma}(\mathbf{V}^T\zeta) - \boldsymbol{\sigma}_{\mathrm{v}}(\mathbf{V}^T\zeta)\mathbf{V}^T\zeta\right) - k_{\mathrm{w}}\mathbf{w}\right) \\
\dot{\mathbf{V}} &= \gamma_{\mathrm{V}}\left(\rho\,\zeta\mathbf{w}^T\boldsymbol{\sigma}_{\mathrm{v}}(\mathbf{V}^T\zeta) - k_{\mathrm{V}}\mathbf{V}\right),
\end{aligned}
\qquad (40)
$$

where γ_{w} and γ_{V} are learning coefficients, and k_{w} and k_{V} are σ-modification gains.

Remark 1 It will be shown in Section 4 that the derivate of Lyapunov function is negative outside a compact set. In this case, to avoid the persistence excitation condition of inputs to the NN and to guarantee boundedness of $\tilde{\mathbf{w}}$ and $\tilde{\mathbf{V}}$, the σ-modification terms was considered in the tuning rules given in (40) [20, 25].

Lemma 3 The approximation error of the NN has the following upper bound:

$$
\left|\Delta(\mathbf{z},\boldsymbol{\eta},u) - u_{ad}\right| \le 2\sqrt{m}\, M_{\mathrm{w}} + \varepsilon_{2M} \qquad (41)
$$

Proof Using (33) and (35), it can be written

$$
\begin{aligned}
\left|\Delta(\mathbf{z},\boldsymbol{\eta},u) - u_{ad}\right| &= \left|\mathbf{w}^{*T}\boldsymbol{\sigma}\left(\mathbf{V}^{*T}\zeta\right) + \varepsilon_2 - \mathbf{w}^T\boldsymbol{\sigma}\left(\mathbf{V}^T\zeta\right)\right| \\
&\le \left|\mathbf{w}^{*T}\boldsymbol{\sigma}\left(\mathbf{V}^{*T}\zeta\right)\right| + |\varepsilon_2| + \left|\mathbf{w}^T\boldsymbol{\sigma}\left(\mathbf{V}^T\zeta\right)\right| \\
&\le \left\|\mathbf{w}^{*T}\right\|\cdot\left\|\boldsymbol{\sigma}\left(\mathbf{V}^{*T}\zeta\right)\right\| + |\varepsilon_2| + \left\|\mathbf{w}^T\right\|\cdot\left\|\boldsymbol{\sigma}\left(\mathbf{V}^T\zeta\right)\right\|
\end{aligned}
$$

Next, considering the facts that $\|\mathbf{w}\| \le M_{\mathrm{w}}$, $|\varepsilon_2| \le \varepsilon_{2M}$, and $\|\boldsymbol{\sigma}\| \le \sqrt{m}$, it can be obtained

$$|\Delta(\mathbf{z}, \boldsymbol{\eta}, u) - u_{ad}| \le 2\sqrt{m}\, M_{\mathrm{w}} + \varepsilon_{2M}\,. \qquad \square$$

3.5 Adaptive robustifying controller

Using the neuro-adaptive control term u_{ad} with adaptation rules in (40), the matched uncertainty Δ cannot be completely compensated for and there exists approximation error $\delta(t)$, which can increase the ultimate error bound. Moreover, the error ρ_a can also have the same effect on the ultimate error bound. In order to compensate for these errors, an adaptive robustifying control term u_R is incorporated into the control law.

First, consider that the upper bound on the NN approximation error plus the compensable part of ρ_a can be derived using (37) and (41) as

$$\begin{aligned}
|\delta| + \lambda_1 |\Delta - u_{ad}| &\le \left(\varepsilon_{2M} + 2\sqrt{m}M_{\mathrm{w}}\right) + \alpha M_{\mathrm{V}} \left\|\mathbf{w}^* - \mathbf{w}\right\| \|\zeta\| \\
&\quad + \alpha M_{\mathrm{w}} \left\|\mathbf{V}^* - \mathbf{V}\right\|_{\mathrm{F}} \|\zeta\| + \lambda_1 \left(2\sqrt{m}\, M_{\mathrm{w}} + \varepsilon_{2M}\right) \\
&\le \left(\varepsilon_{2M} + 2\sqrt{m}M_{\mathrm{w}}\right)(1+\lambda_1) + \alpha M_{\mathrm{V}} \left\|\mathbf{w}^*\right\| \|\zeta\| \\
&\quad + \alpha M_{\mathrm{V}} \|\mathbf{w}\| \|\zeta\| + \alpha M_{\mathrm{w}} \left\|\mathbf{V}^*\right\|_{\mathrm{F}} \|\zeta\| + \alpha M_{\mathrm{w}} \|\mathbf{V}\|_{\mathrm{F}} \|\zeta\| \\
&\le \left(\varepsilon_{2M} + 2\sqrt{m}M_{\mathrm{w}}\right)(1+\lambda_1) + \alpha M_{\mathrm{V}}\, M_{\mathrm{w}} \|\zeta\| \\
&\quad + \alpha M_{\mathrm{V}} \|\mathbf{w}\| \|\zeta\| + \alpha M_{\mathrm{w}}\, M_{\mathrm{V}} \|\zeta\| + \alpha M_{\mathrm{w}} \|\mathbf{V}\|_{\mathrm{F}} \|\zeta\| \\
&\le \varphi^* \left(1 + \|\zeta\| \left(1 + \|\mathbf{V}\|_{\mathrm{F}} + \|\mathbf{w}\|\right)\right) = \varphi^* \chi,
\end{aligned} \tag{42}$$

where $0 < \lambda_1 < 1$ is the same as in (19), and

$$\begin{aligned}
\varphi^* &\triangleq \max\left\{(1+\lambda_1)\left(\varepsilon_{2M} + 2\sqrt{m}M_{\mathrm{w}}\right), \alpha M_{\mathrm{w}}, \alpha M_{\mathrm{V}}, 2\alpha M_{\mathrm{V}}\, M_{\mathrm{w}}\right\}, \\
\chi &\triangleq 1 + \|\zeta\| \left(1 + \|\mathbf{V}\|_{\mathrm{F}} + \|\mathbf{w}\|\right).
\end{aligned}$$

Hence, $|\delta| + \lambda_1 |\Delta - u_{ad}|$ is bounded to the multiplication of the known function χ and the unknown gain φ^*.

Now, the following adaptive robustifying control term is introduced:

$$u_R = \frac{1}{1-\lambda_1} \varphi\, \chi\, \mathrm{sign}(\rho), \tag{43}$$

with the following adaptation rule:

$$\dot{\varphi} = \gamma_\varphi \, \chi |\rho| , \tag{44}$$

where φ denotes estimation of the unknown gain φ^* and γ_φ is the adaptation coefficient.

Remark 2 It is well known that due to the universal approximation property of NNs, the approximation error is bounded. Hence, it is always possible to find a positive constant U_M such that

$$|u_R| \le U_M \tag{45}$$

4 Stability Analysis

In this section, it is shown that the closed-loop error dynamics are ultimately bounded. The analysis is based on substituting the state vector $\boldsymbol{\xi}$ with the SPR augmented error ρ_{ag} using the LKY lemma.

DEFINITION: Let Ω_Δ be the compact set defined in (33) and Ω_{r_Δ} be the largest hypersphere in the error space $\mathbf{E} = \left[\boldsymbol{\xi}, \|\tilde{\mathbf{w}}\|, \|\tilde{\mathbf{V}}\|_F, |\tilde{\varphi}| \right]$, defined as

$$\Omega_{r_\Delta} \triangleq \left\{ \mathbf{E} \,\middle|\, \|\mathbf{E}\| \le r_\Delta \right\} \tag{46}$$

such that for every $\mathbf{E} \in \Omega_{r_\Delta}$ there exists $(\mathbf{z}, \boldsymbol{\eta}, u) \in \Omega_\Delta$, where r_Δ is a positive number.

ASSUMPTION 4: Assume that the following inequality holds:

$$r < \sqrt{S_m / S_M}\, r_\Delta , \tag{47}$$

where S_m and S_M are the minimum and the maximum eigenvalues of the following matrix, respectively:

$$\mathbf{S} = \frac{1}{2} \begin{bmatrix} \mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma_{\mathbf{w}}^{-1} & 0 & 0 \\ \mathbf{0} & 0 & \gamma_{\mathbf{V}}^{-1} & 0 \\ \mathbf{0} & 0 & 0 & \gamma_\varphi^{-1} \end{bmatrix} .$$

r will be defined in the proof of the following Theorem.

Theorem *Consider the linear controller in (11), the neuro-adaptive control term* u_{ad} *in (35) with adaptation rules in (40), and the robustifying control term* u_R *in (43) with adaptation rules in (44). If Assumptions 1- 4 hold and the initial error* ***E****(0) belongs to the compact set in (46), then the errors* ξ, $\tilde{\mathbf{w}}$, *and* $\tilde{\mathbf{V}}$ *in the closed-loop system are uniformly ultimately bounded.*

Proof Let candidate the following Lyapunov function:

$$L = \frac{1}{2}\xi^T \mathbf{P}\xi + \frac{1}{2\gamma_{\mathbf{w}}}\|\tilde{\mathbf{w}}\|^2 + \frac{1}{2\gamma_{\mathbf{V}}}\|\tilde{\mathbf{V}}\|_{\mathrm{F}}^2 + \frac{1}{2\gamma_{\varphi}}|\tilde{\varphi}|^2, \tag{48}$$

where the symmetric matrix $\mathbf{P}$ is the unique positive-definite solution of (20) and $\tilde{\varphi}$ is defined as $\tilde{\varphi} \triangleq \varphi^* - \varphi$, where φ^* and φ are the same as before. Moreover, recall that $\mathbf{w}^*$ and $\mathbf{V}^*$ are the ideal constant weights for the NN defined in (34). Then, Eq. (37) yields $\dot{\mathbf{w}} = -\dot{\tilde{\mathbf{w}}}$ and $\dot{\mathbf{V}} = -\dot{\tilde{\mathbf{V}}}$.

Using (15), the time-derivative of (48) becomes

$$\begin{aligned}\dot{L} = &-\frac{1}{2}\xi^{\mathbf{T}}\mathbf{Q}\xi + \xi^T \mathbf{P}\mathbf{b}_{cl}\left(\Delta - u_{ad} - u_R\right) + \xi^T \mathbf{P}\mathbf{G}_{cl}\boldsymbol{\Delta}_{\boldsymbol{\eta}} \\ &-\frac{1}{\gamma_{\mathbf{w}}}\tilde{\mathbf{w}}^T\dot{\mathbf{w}} - \frac{1}{\gamma_{\mathbf{V}}}\mathrm{tr}\left(\tilde{\mathbf{V}}^T\dot{\mathbf{V}}\right) - \frac{1}{\gamma_{\varphi}}\tilde{\varphi}\dot{\varphi}.\end{aligned} \tag{49}$$

Using (18), (21) can be written as

$$\xi^T \mathbf{P}\mathbf{b}_{cl} = \rho_{ag} = \rho + \rho_a\,. \tag{50}$$

By substituting (36) and (50) into (49), $\dot{L}$ becomes

$$\begin{aligned}\dot{L} = &-\frac{1}{2}\xi^{\mathbf{T}}\mathbf{Q}\xi + \rho\left(\tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma} - \dot{\boldsymbol{\sigma}}\mathbf{V}^T\zeta\right) + \mathrm{tr}\left(\tilde{\mathbf{V}}^T\zeta\mathbf{w}^T\dot{\boldsymbol{\sigma}}\right) + \delta + u_R\right) \\ &-\frac{1}{\gamma_{\mathbf{w}}}\tilde{\mathbf{w}}^T\dot{\mathbf{w}} - \frac{1}{\gamma_{\mathbf{V}}}\mathrm{tr}\left(\tilde{\mathbf{V}}^T\dot{\mathbf{V}}\right) + \rho_a\left(\Delta - u_{ad} - u_R\right) - \frac{1}{\gamma_{\varphi}}\tilde{\varphi}\dot{\varphi} + \xi^T\mathbf{P}\mathbf{G}_{cl}\boldsymbol{\Delta}_{\boldsymbol{\eta}}.\end{aligned} \tag{51}$$

Using (14), the upper bound of the modelling error, defined in (3), can be represented as

$$\left\|\boldsymbol{\Delta}_{\boldsymbol{\eta}}(\mathbf{z},\boldsymbol{\eta})\right\| \le d_0 + \left(d_1 + d_2\right)\|\mathbf{x}\| \le d_0 + \left(d_1 + d_2\right)\left(\|\xi\| + \|\mathbf{x}_l\|\right). \tag{52}$$

Since the closed-loop reference model in (12) is stable, it is always possible to find a positive constant d_3, which satisfies $\|\mathbf{x}_l\| \le d_3$; then, substituting this into (52) yields

$$\left\|\boldsymbol{\Delta}_{\boldsymbol{\eta}}(\mathbf{z},\boldsymbol{\eta})\right\| \le \alpha_0 + \alpha_1 \|\boldsymbol{\xi}\|, \quad \alpha_0 = d_0 + \alpha_1 d_3, \quad \alpha_1 = d_1 + d_2. \tag{53}$$

Substituting (53) and the robustifying control term (43) into (51) gives

$$\begin{aligned} \dot{L} \le & -\frac{1}{2} Q_m \|\boldsymbol{\xi}\|^2 + \tilde{\mathbf{w}}^T \left(\rho\left(\boldsymbol{\sigma} - \dot{\boldsymbol{\sigma}} \mathbf{V}^T \boldsymbol{\zeta}\right) - k_{\mathrm{w}} \mathbf{w} - \frac{1}{\gamma_{\mathrm{w}}} \dot{\mathbf{w}} \right) + k_{\mathrm{w}} \tilde{\mathbf{w}}^T \mathbf{w} \\ & + \mathrm{tr}\left(\tilde{\mathbf{V}}^T \left(\rho \boldsymbol{\zeta} \mathbf{w}^T \dot{\boldsymbol{\sigma}} - k_{\mathrm{V}} \mathbf{V} - \frac{1}{\gamma_{\mathrm{V}}} \dot{\mathbf{V}} \right) \right) + k_{\mathrm{V}}\, \mathrm{tr}\left(\tilde{\mathbf{V}}^T \mathbf{V} \right) + |\rho||\delta| - \frac{\varphi \chi |\rho|}{1-\lambda_1} \\ & + \lambda_1 |\rho| |\Delta - u_{ad}| + \lambda_1 \frac{|\rho| \varphi \chi}{1-\lambda_1} + \lambda_0 \beta_M - \frac{1}{\gamma_\varphi} \tilde{\varphi} \dot{\varphi} + \|\mathbf{P}\mathbf{G}_{cl}\| \|\boldsymbol{\xi}\| \left(\alpha_0 + \alpha_1 \|\boldsymbol{\xi}\| \right), \end{aligned} \tag{54}$$

where Q_m denotes the smallest eigenvalue of the matrix $\mathbf{Q}$ and $\beta_M \triangleq 2\sqrt{m} M_{\mathrm{w}} + \varepsilon_{1M} + U_M$ is the upper bound of $|\Delta - u_{ad} - u_R|$ that can be derived using (41) and (45) .

Using the adaptation rules in (40) and the bound defined in (42), $\dot{L}$ becomes

$$\begin{aligned} \dot{L} \le & -\left(\frac{1}{2} Q_m - \alpha_1 \|\mathbf{P}\mathbf{G}_{cl}\| \right) \|\boldsymbol{\xi}\|^2 - k_{\mathrm{w}} \|\tilde{\mathbf{w}}\|^2 - k_{\mathrm{V}} \left\|\tilde{\mathbf{V}}\right\|^2 + k_{\mathrm{w}} M_{\mathrm{w}} \|\tilde{\mathbf{w}}\| \\ & + k_{\mathrm{V}} M_{\mathrm{V}} \left\|\tilde{\mathbf{V}}\right\| + \left(|\rho| \chi \left(\varphi^* - \varphi \right) - \frac{1}{\gamma_\varphi} \tilde{\varphi} \dot{\varphi} \right) + \lambda_0 \beta_M + \alpha_0 \|\mathbf{P}\mathbf{G}_{cl}\| \|\boldsymbol{\xi}\|. \end{aligned} \tag{55}$$

Using the adaptation rule in (44) and completion of the square terms gives

$$\dot{L} \le -A_\xi \|\boldsymbol{\xi}\|^2 - \left(k_{\mathrm{w}} - 1\right) \|\tilde{\mathbf{w}}\|^2 - \left(k_{\mathrm{V}} - 1\right) \left\|\tilde{\mathbf{V}}\right\|^2 + R, \tag{56}$$

where

$$\begin{aligned} A_\xi & \triangleq \left(\frac{1}{2} Q_m - \alpha_1 \|\mathbf{P}\mathbf{G}_{cl}\| - 1 \right) \\ R & \triangleq \frac{\left(k_{\mathrm{w}} M_{\mathrm{w}}\right)^2}{4} + \frac{\left(k_{\mathrm{V}} M_{\mathrm{V}}\right)^2}{4} + \frac{\left(\alpha_0 \|\mathbf{P}\mathbf{G}_{cl}\|\right)^2}{4} + \lambda_0 \beta_M \end{aligned} \tag{57}$$

Let α_1, defined in (53), be sufficiently small such that matrix $\mathbf{Q}$ ensures the following condition:

$$\lambda_{\min}(\mathbf{Q}) > 2\alpha_1 \|\mathbf{P}\mathbf{G}_{cl}\| + 2\,. \tag{58}$$

Moreover, let $k_{\mathrm{V}} > 1$ and $k_{\mathrm{w}} > 1$.

Now, let define the following compact set around the origin

$$\Omega_r \triangleq \left\{ \mathbf{E} \,\middle|\, \|\mathbf{E}\| \le r \right\}, \tag{59}$$

where

$$r \triangleq \max\left(\sqrt{\frac{R}{A_\xi}}, \sqrt{\frac{R}{k_w - 1}}, \sqrt{\frac{R}{k_V - 1}}\right). \tag{60}$$

Equation (56) shows that $\dot{L} \le 0$ once the errors are outside the compact set Ω_r. Now, consider the Lyapunov function (48), which can alternatively be written as $L = \mathbf{E}^T\mathbf{S}\mathbf{E}$, where $\mathbf{E}$ and $\mathbf{S}$ are the same as given before and $S_m \|\mathbf{E}\|^2 \le L(\mathbf{E}) \le S_M \|\mathbf{E}\|^2$, in which S_m and S_M are the smallest and the largest eigenvalues of **S**, respectively. Let L_M be the maximum value of the Lyapunov function L on the boundary of Ω_r: $L_M = S_M r^2$ and L_m be its minimum value on the boundary of Ω_{r_Δ} : $L_m = S_m r_\Delta^2$. If $L_M < L_m$ or equivalently Assumption 4 holds, then it can be concluded that $\Omega_r \subset \Omega_{r_\Delta}$. This ensures that if initially the error is inside Ω_{r_Δ} (i.e. $\mathbf{E}(0) \in \Omega_{r_\Delta}$), then according to the standard Lyapunov theorem extension, the error trajectory $\mathbf{E}(t)$ is ultimately bounded [19, 26]. □

Corollary The system output y tracks the reference signal y_d with bounded error trajectory.

Proof Consider the following compact set:

$$\Omega_\xi \triangleq \left\{\xi \middle| \|\xi\| \le \sqrt{\frac{R}{A_\xi}}\right\}.$$

From (56), it can be seen that $\dot{L}$ is strictly negative as long as ξ is outside the compact set Ω_ξ. Therefore, there exists a constant time T such that for $t > T$ the error vector $\xi = \mathbf{x}_l - \mathbf{x}$ converges to Ω_ξ [19]. This means that the error is bounded to $\|\xi\| \le \sqrt{R/A_\xi}$. Moreover, all signals in the reference model (12) are bounded and $(y_d - y_l) \to 0$ as $t \to \infty$ according to Assumption 2. This ensures that $(y_d - y) \to \varepsilon$ as $t \to \infty$, where $\varepsilon \le \sqrt{R/A_\xi}$. □

Remark 3 When the exact value of coefficient matrices $\mathbf{m}$, $\mathbf{n}$, $\mathbf{F}$ and $\mathbf{G}$ in the expanded model (2) are not available (e.g. due to parameters uncertainty), the estimated values $\hat{\mathbf{m}}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$ may be used to design the linear controller. In this case, some modelling errors (arise from the difference between the real and the estimated value of coefficients) can be embodied in Δ and $\mathbf{\Delta_\eta}$, while assumptions 1 and 3 are satisfied. Moreover, it is important to point out that in order to design the controller, it is not necessary to have a normal form of the plant as in (1), and only a linear approximation, which satisfies assumptions 1, 2 and 3, suffices.

Remark 4 It will be shown in the followings that conditions (47) and (58) can be satisfied easier by increasing the rate of convergence of the error dynamic. Let define new state variables as $\varsigma = \mathbf{T}^{-1}\xi$ where $\mathbf{T}$ is defined such $\mathbf{\Lambda}_{cl} = \mathbf{T}^{-1}\mathbf{A}_{cl}\mathbf{T}$ represents a diagonal matrix. Replacing ξ by ς in (15) and repeating the proof of the Theorem, results in a similar condition as in (58)

$$\lambda_{\min}(\mathbf{Q}') > 2\alpha_1 \|\mathbf{T}\| \|\mathbf{P}'\mathbf{G}'_{cl}\| + 2, \qquad (61)$$

where $\mathbf{Q}' = \mathbf{T}^T\mathbf{Q}\mathbf{T}$, $\mathbf{P}' = \mathbf{T}^T\mathbf{P}\mathbf{T}$ and $\mathbf{G}'_{cl} = \mathbf{T}^{-1}\mathbf{G}_{cl}$.

From (13), $\|\mathbf{G}_{cl}\| = \sqrt{\lambda_{\max}\left(\mathbf{G}_{cl}^T\mathbf{G}_{cl}\right)} = 1$ and $\mathbf{P}' = \mathbf{P}'^T$. Hence,

$$\|\mathbf{P}'\mathbf{G}'_{cl}\| \le \|\mathbf{P}'\| \|\mathbf{T}^{-1}\| \|\mathbf{G}_{cl}\| = \lambda_{\max}(\mathbf{P}') \|\mathbf{T}^{-1}\|. \qquad (62)$$

On the other hand, one can conclude that $\lambda_{\max}(\mathbf{\Lambda}_{cl} + \mathbf{\Lambda}_{cl}^T) < 0$. Therefore, [27]

$$\lambda_{\max}(\mathbf{P}') \le \frac{\lambda_{\max}(\mathbf{Q}')}{\left|\lambda_{\max}(\mathbf{\Lambda}_{cl} + \mathbf{\Lambda}_{cl}^T)\right|}. \qquad (63)$$

Substituting (62) and (63) into (61), it can be represented in a new form as

$$\lambda_{\min}(\mathbf{Q}') > \frac{2\alpha_1 \|\mathbf{T}^{-1}\| \|\mathbf{T}\| \lambda_{\max}(\mathbf{Q}')}{\left|\lambda_{\max}(\mathbf{\Lambda}_{cl} + \mathbf{\Lambda}_{cl}^T)\right|} + 2 \qquad (64)$$

As Eq. (64) implies, condition (58) or equivalently (61) may be satisfied with smaller value of $\lambda_{\min}(\mathbf{Q}')$ for a fixed value of α_1, by increasing the rate of convergence of the error dynamics (i.e. acquiring larger eigenvalues for $\mathbf{\Lambda}_{cl}$ or alternatively for $\mathbf{A}_{cl}$). This can be achieved by designing suitable controller in (11). Moreover, in the 7th step of the optimization algorithm (presented in Appendix), the constrain may be satisfied by smaller values for $\lambda_{\min}(\mathbf{Q}')$; this yields smaller values for J and consequently $\|\mathbf{c}_a\|$, which is the output vector of the unknown error dynamics ρ_a that is derived in the optimization algorithm given in Appendix. Hence, it decreases $\lambda_0 = \sup_t\left(|\rho_a(t)|\right)$. Consequently, according to (57) and (60), the error bound r decreases, which helps to satisfy Assumption 4.

Remark 5 The definition of R in (57) shows that the ultimate error bound mainly depends on the unmatched uncertainty $\left(\alpha_0 \|\mathbf{PG}_{cl}\|\right)^2/4$ and the multiplication of the NN approximation error bound (β_M) and λ_0. As it is stated in Remark 4, acquiring larger eigenvalues for $\mathbf{\Lambda}_{cl}$ (or alternatively for $\mathbf{A}_{cl}$) helps to achieve smaller values for λ_0, which consequently reduces R and r. On the other hand, when the NN begins to learn, the upper bound of the NN approximation error (β_M) will decrease; consequently $\lambda_0\beta_M$ decreases. However, the term $\left(\alpha_0 \|\mathbf{PG}_{cl}\|\right)^2/4$ always exists and cannot be reduced by the control design. In fact, in order to compensate effects of the unmatched uncertainty, it is necessary to design a special reference signal like $y_d(\boldsymbol{\eta}, \mathbf{z})$ (similar to the backstepping approach). However, in this case, only the stability of the system may be achieved and the tracking ability will be lost. This drawback is shared by the NN control methods in which the reconstruction error of the NN is not compensated for because the NN tracking problem of nonlinear non-minimum phase systems is due to the existence of the unmatched uncertainty or the modelling error of internal dynamics.

Remark 6 As Eq. (57) shows, the unmatched uncertainties and the auxiliary error term ρ_a (embodied in constants α_0 and λ_0) can increase the error bound. If $\alpha_0 = 0$ and $\lambda_0 = 0$, then by setting $k_{\mathrm{w}} = k_{\mathrm{V}} = 0$ in the adaptation rules (40) asymptotic stability can be achieved. In this case, since $L > 0$ and $\dot{L} \le 0$ the boundedness of adaptive weights $\tilde{\mathbf{w}}, \tilde{\mathbf{V}}$ and $\tilde{\varphi}$ (and hence $\mathbf{w}, \mathbf{V}$ and φ) can be directly concluded.

Remark 7 When a discontinuous input signal is applied to a system, the chattering phenomenon appears. Many methods have been proposed in literatures to reduce the chattering including continuous approximation of the discontinuous input signal. A continuous approximation of $\mathrm{sgn}(\rho)$ in (43) is $\tanh(\rho/\varepsilon)$ or alternatively $\rho/(|\rho|+\delta)$ where $\varepsilon > 0$ and $0 < \delta < 1$. However, using these continuous functions might increase the ultimate error bound proportional to ε and δ.

The block diagram of the proposed control method is depicted in Fig. 1.

5 Simulation Example

Performance of the proposed controller is shown through simulations using the TORA system, which is a nonlinear non-minimum phase system. First, stabilization of the system is performed in order to compare the results with the backstepping method, which has been proposed in reference [14]. Then, the tracking of a reference signal is sought for this system.

This system, depicted in Fig. 2, is described by the following equations [8, 14]:

$$\begin{cases}(M+m)\ddot{x}+ml(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)=-kx\\(J+ml^2)\ddot{\theta}+ml\cos\theta\,\ddot{x}=\tau,\end{cases}$$

where θ is the angle of rotation, x is the translational displacement, and τ is the control torque. The positive constants k, l, J, M, and m denote the spring stiffness, the radius of rotation, the moment of inertia, the mass of the cart, and the eccentric mass, respectively.

Let define the states and the input variables as

$$\eta_1 = x+\frac{ml}{M+m}\sin\theta,\quad \eta_2 = \dot{x}+\frac{ml}{M+m}\dot{\theta}\cos\theta,\quad z_1=\theta,\quad z_2=\dot{\theta},\quad u=\tau$$

In these coordinates, the system can be described by a set of equations in the form of (1) as

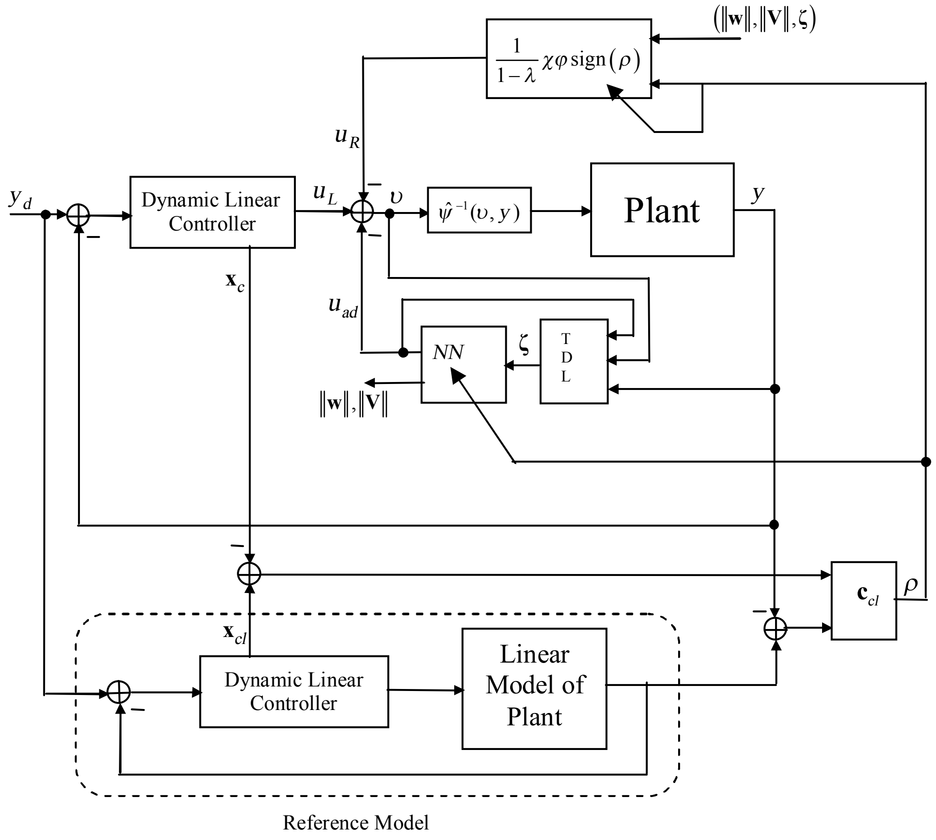
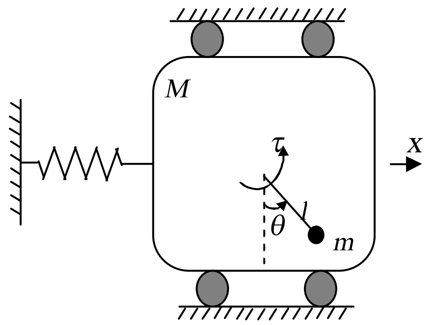


Fig. 1 Block diagram of the proposed control method

Fig 2. A Translational Oscillator with Rotational Actuator (TORA)



$$
\begin{cases}
\dot{z}_1 = z_2 \\
\dot{z}_2 = ka_1\phi^{-1}(z_1)\cos z_1\eta_1 - a_1^2 a_2\phi^{-1}(z_1)\sin z_1 \cos z_1 \\
\qquad - m^2 l^2 \phi^{-1}(z_1) z_2^2 \sin z_1 \cos z_1 + (M+m)\phi^{-1}(z_1)\, u \\
\dot{\eta}_1 = \eta_2 \\
\dot{\eta}_2 = -a_2\eta_1 + a_3 \sin z_1,
\end{cases}
$$

where

$$
\phi(z_1) = (M+m)(J+ml^2) - m^2 l^2 \cos^2\theta,
$$

$$
a_1 = ml, \quad a_2 = \frac{k}{M+m} \quad \text{and} \quad a_3 = \frac{k\,ml}{(M+m)^2}.
$$

The zero dynamics of this system are

$$\begin{cases}\dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = -a_2\eta_1.\end{cases}$$

It is straightforward to check that these zero dynamics are unstable, so the system is non-minimum phase [8]. Let the available linearised version of the TORA system be written as

$$\begin{cases}\dot{z}_1 = z_2 \\ \dot{z}_2 = -\hat{a}_1^2\hat{a}_2\hat{\phi}^{-1}(0)z_1 + \hat{k}\hat{a}_1\hat{\phi}^{-1}(0)\eta_1 + (M+\hat{m})\phi^{-1}(0)u_L \\ \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = -\hat{a}_2\eta_1 + \hat{a}_3 z_1,\end{cases}$$

where $\hat{m}, \hat{k}, \hat{J}$ and consequently $\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{\phi}$ are the estimates of parameters m, k, J and a_1, a_2, a_3, ϕ, respectively. Hence, uncertain parts of the model can be represented as

$$\begin{aligned}\psi(\boldsymbol{\eta}, \mathbf{z}, u) = \frac{1}{(M+\hat{m})\hat{\phi}^{-1}(0)} & \Big\{(M+m)\phi^{-1}(z_1)u + \left[\alpha_1\phi^{-1}(z_1)\cos z_1 - \hat{\alpha}_1\hat{\phi}^{-1}(0)\right]\eta_1 \\ & - m^2 l^2\phi^{-1}(z_1)z_2^2\sin z_1\cos z_1 \\ & - \left[\alpha_1^2\alpha_2\phi^{-1}(z_1)\sin z_1\cos z_1 - \hat{\alpha}_1^2\hat{\alpha}_2\hat{\phi}^{-1}(0)z_1\right]\Big\}, \\ \Delta_\eta(\boldsymbol{\eta}, \mathbf{z}) = -(a_2 - \hat{a}_2)\eta_1 + a_3\sin z_1 - \hat{a}_3 z_1. &\end{aligned}$$

Let consider the best available approximation of ψ as $\hat{\psi} = \upsilon = cu$, where c should be selected such that Assumption 3 holds; that is

$$c \geq 0.5\frac{(M+m)\phi^{-1}(z_1)}{(M+\hat{m})\hat{\phi}^{-1}(0)} > 0 \qquad \forall z_1 \in \Omega_z.$$

To ensure that this condition holds for $\hat{m} < 2m$ and $\hat{J} < 2J$, it is assumed that $c = 1$.

The linear controller is designed using the pole placement approach by solving the Diophantine equation [21]. The state space form of this controller for the linear system without parameters uncertainties is

$$\dot{x}_c = \begin{bmatrix} -15.6 & 1 & 0 & 0 \\ -2068 & -198 & 67932 & -4529 \\ 0.28 & 0 & 0 & 1 \\ 10.8 & 0 & -127.8 & 0 \end{bmatrix} x_c + \begin{bmatrix} 15.6 \\ 121.7 \\ -0.28 \\ -10.4 \end{bmatrix} (y_d - y)$$

$$u_L = \begin{bmatrix} 1.03 & 0.1 & -35.4 & 2.4 \end{bmatrix} x_c$$

The NN is of MLP type and has 10 neurons in one hidden layer with tangent hyperbolic activation functions. The weights are initialised randomly using small numbers. The input vector to the NN for $n_1 = 4 \geq n$ is

$$\zeta = [1, y(t), y(t-T_d), y(t-2T_d), y(t-3T_d), u_\alpha(t), u_\alpha(t-T_d), u_{ad}(t-T_d)]^T,$$

with $T_d = 10$ ms. The learning coefficients are selected as $\gamma_w = \gamma_V = 2$, $\gamma_\varphi = 0.05$, and $k_w = k_V = 1.2$. The vector $\mathbf{c}_{ag}$ is designed using the algorithm given in Appendix as

$$\mathbf{c}_{ag} = 10^3 \times [-0.842, 2.322, -0.080, 22.160, -1.021, 2.311, -100.345, 21.430]$$

For the sake of comparison, the simulations are carried out using the same parameters and initial conditions as in reference [14]

$J = 0.0002175$ kg/m^2, $M = 1.3608$ kg,

$m = 0.096$ kg, $l = 0.0592$ m, and $k = 186.3$ N/m.

$$\eta_1(0) = 0.025,\ \eta_2(0) = z_1(0) = z_2(0) = 0.$$

Moreover, the physical constrain on the control effort is considered to be $|\tau| \le 0.1$ N.m as in [14].

Simulation results are depicted in Figs. 3–8. First, performance of the closed-loop system is evaluated without parameters uncertainties, namely $\hat{m} = m$, $\hat{k} = k$, and $\hat{J} = J$. In this case, the proposed controller is used only as a stabilizer with $y_d = 0$.

Fig. 3 shows response of the closed-loop system in original coordinates x and θ, and the control torque τ. Observe that the convergence rate of the proposed approach is almost the same as the backstepping-based controller.

To verify the robustness of the proposed controller, simulations are repeated in presence of parameters uncertainties with $\hat{m} = 1.1m$, $\hat{k} = 1.15k$, and $\hat{J} = 1.15J$. Notice that theses uncertainties can be embedded in Δ and since the proposed approach compensates Δ adaptively (see Fig. 5a) the approach is robust against parameters uncertainties. Simulation results, presented in Fig. 4, confirm the robustness of the closed-loop system. As this figure shows, the proposed controller stabilizes the system very well and the state variables converge to zero while the backstepping approach cannot stabilize the state variables, especially for the angular movement θ. Moreover, the control signal of the backstepping method has very sharp pulses, which may damage the actuator of the system. This is mainly because the backstepping approach is model based, while the proposed method requires only an approximate linear model of the system. However, it should be mentioned that the backstepping approach [14] assures global stability of the system while the proposed approach provides semi-global stability in the sense that it is local with respect to the NN approximation domain Ω_{Δ} .

The validity of the bounds on the auxiliary error ρ_a and the matched uncertainties Δ_{η} can be examined in Figs. 5b and 5c, respectively. Moreover, the norms of the adaptive weights are depicted in Fig. 5d.

Next, the ability of the proposed controller in tracking a non-zero reference signal is investigated. Fig. 6 shows the system response in absence of parameters uncertainties when u_L and $u_L - u_{ad} - u_R$ are applied separately. It is clear that the system response in both cases is satisfactory except for some oscillations at the beginning.

On the other hand, as Fig. 7 shows, when there are parameter uncertainties, the proposed controller has acceptable response, while the linear controller cannot control the system and becomes unstable. In addition, effect of the robustifying control term u_R on the convergence rate and the ultimate error bound can be observed from Fig. 8.

6 Conclusions

A direct adaptive output-feedback control method for non-minimum phase nonlinear systems was proposed in this paper. The proposed method does not relay on state estimation with the aid of introducing a SPR augmented error signal using the LKY lemma. The approach can be applied to non-affine nonlinear systems, which have partially known Lipschitz continuous functions in their arguments with sufficiently small zero dynamic modelling error. The ultimate boundedness of the tracking error as well as boundedness of the NN weights was shown using the Lyapunov direct method. Simulation results, performed on the TORA system, showed good performance of the proposed method as compared to the backstepping method.

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Appendix:

In this appendix, an optimization algorithm is proposed to find appropriate values for $\mathbf{c}_{cl}$ and $\mathbf{c}_a$, i.e. the output vectors for the available and the unavailable error dynamics, given in (18), respectively.

Let define the objective function as

$$J = \lambda_J \frac{\|\mathbf{c}_a\|}{\|\mathbf{c}_{cl}\|} = \lambda_J \frac{\left(\sum_{i=2}^{n} c_i^2\right)^{\frac{1}{2}}}{\left(c_1^2 + \sum_{k=n+1}^{n+n_c} c_i^2\right)^{\frac{1}{2}}} \tag{A1}$$

where λ_J is a positive constant. The task is to calculate $\mathbf{c}_{cl} = \begin{bmatrix} c_1 & \mathbf{0}_{1\times(n-1)} & c_{n+1} & \cdots & c_{n+n_c} \end{bmatrix}$ and $\mathbf{c}_a = \begin{bmatrix} 0 & c_2 & \cdots & c_n & \mathbf{0} \end{bmatrix}$ such that they minimize (A1) and at the same time, they comply with Eqs. (20) and (21) over $\mathbf{Q}$. The optimization is carried out through the following steps using the gradient decent method:

Step 1: Select $\mathbf{Q} = \mathrm{diag}\begin{pmatrix} q_1 & \cdots & q_{n+n_c} \end{pmatrix} > 0$ where $q_j > 2$ for $j = 1, \ldots, n+n_c$ and solve (20) and (21) for $\mathbf{c}_{ag} = \begin{bmatrix} c_1 & \cdots & c_{n+n_c} \end{bmatrix}$.

Step 2: Give a variation to $q_j + \Delta q_j$ for $j = 1, \ldots, n+n_c$ and derive the corresponding $\mathbf{c}_{ag(j)} = [c_{1j}, \cdots, c_{ij}, \cdots c_{(n+n_c)j}]$ using (20) and (21).

Step 3: Derive $\dfrac{\Delta c_{ij}}{\Delta q_j} = \dfrac{c_{ij} - c_i}{\Delta q_j}$ for $i = 1, \ldots, n+n_c$ and $j = 1, \ldots, n+n_c$.

Step 4: Calculate

$$\frac{\Delta J}{\Delta c_{ij}} = \begin{cases} \dfrac{\lambda_c c_{ij}}{\left(\sum_{k=2}^{n} c_{kj}^2\right)^{\frac{1}{2}} \left(c_{1j}^2 + \sum_{k=n+1}^{n+n_c} c_{kj}^2\right)^{\frac{1}{2}}} & \text{if } 2 \le i \le n \\ -\dfrac{\lambda_c \left(\sum_{k=2}^{n} c_{kj}^2\right)^{\frac{1}{2}}}{\left(c_{1j}^2 + \sum_{k=n+1}^{n+n_c} c_{kj}^2\right)^{\frac{3}{2}}} c_{ij} & \text{otherwise} \end{cases}$$

Step 5: Find $\frac{\Delta J}{\Delta q_j} = \sum_{i=1}^{n+n_c} \frac{\Delta J}{\Delta c_{ij}} \frac{\Delta c_{ij}}{\Delta q_j}$ $\quad j = 1, \ldots, n + n_c$

Step 6: Update matrix **Q** using $q_j(t+1) = q_j(t) - \lambda \frac{\Delta J}{\Delta q_j}$ for $j = 1, \ldots, n + n_c$ and calculate matrix **P** using (20).

Step 7: If $q_j(t+1) \le 2\alpha_1 \|\mathbf{PG}_{cl}\| + 2$, then $q_j(t+1) = q_j(t)$ $(j = 1, \ldots, n + n_c)$.

Step 8: If $\left\| \left(\frac{\Delta J}{\Delta q_1}, \ldots, \frac{\Delta J}{\Delta q_m} \right) \right\| \le \varepsilon_J$ where ε_J is a desired small value, go to Step 9, else $t = (t+1)$ and return to Step 2.

Step 9: Derive $\mathbf{c}_{ag}$ using (20) and (21).

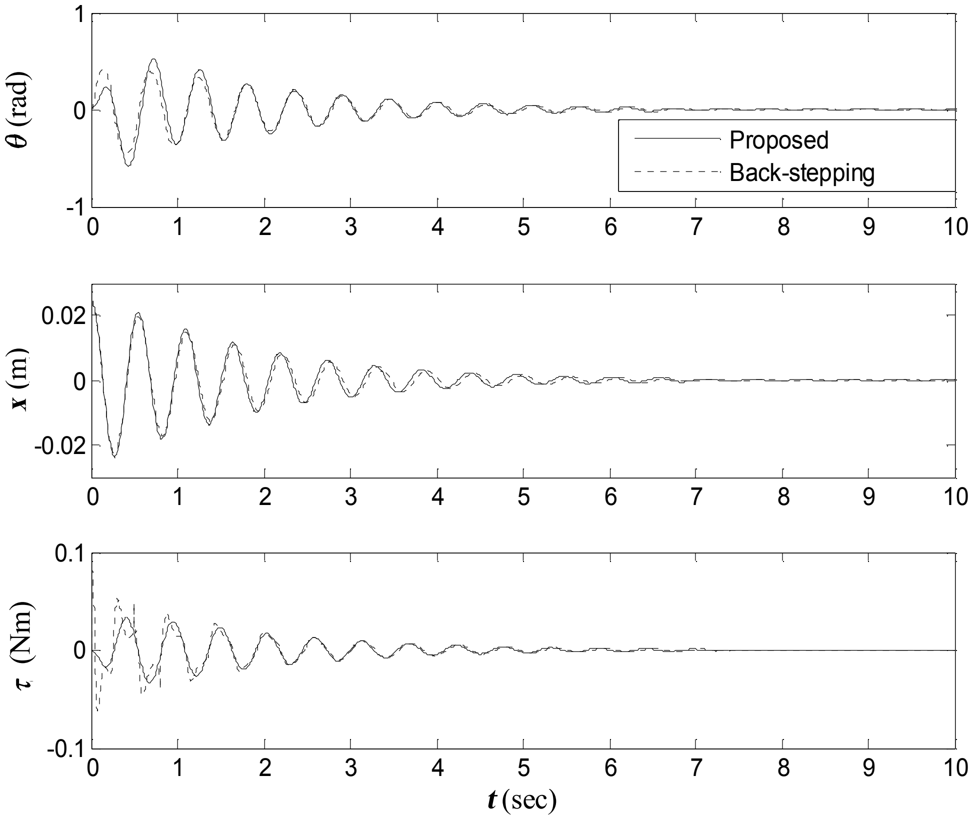

Fig 3. Stabilization of the TORA without parameters uncertainty; solid line: the proposed method, and dotted line: the backstepping controller

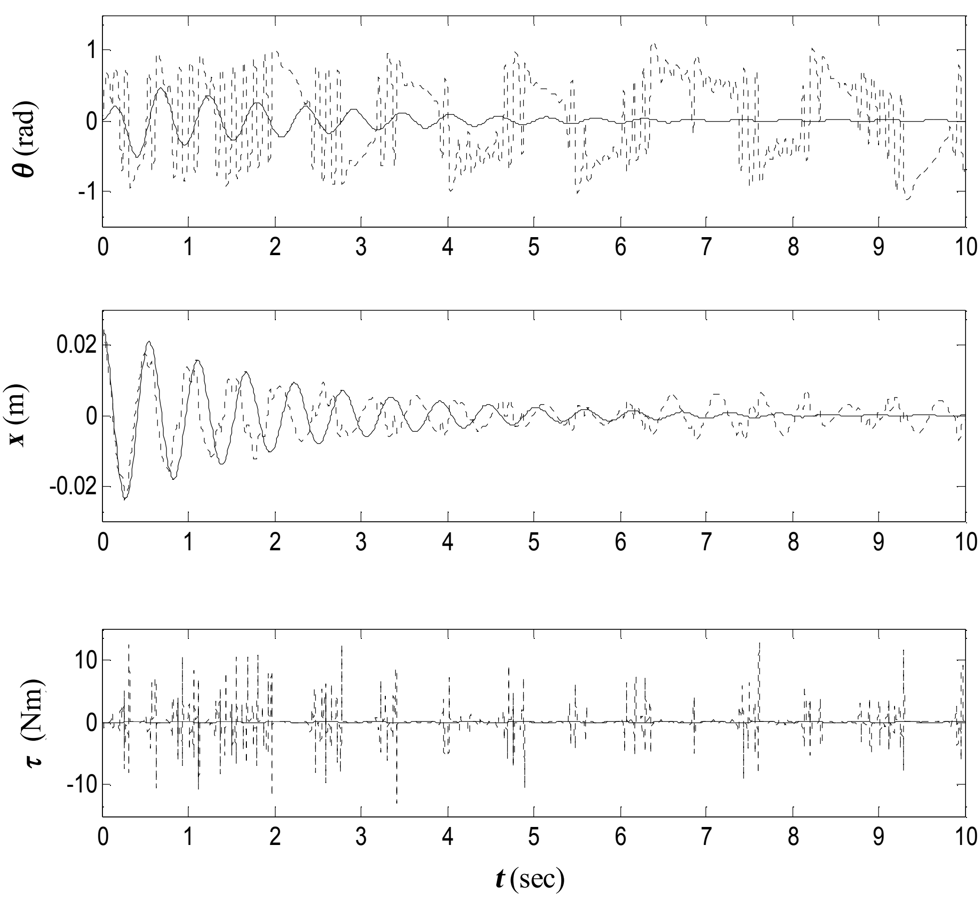

Fig 4. Stabilization of the TORA system with parameters uncertainties; solid line: the proposed method, dotted line: the backstepping approach.

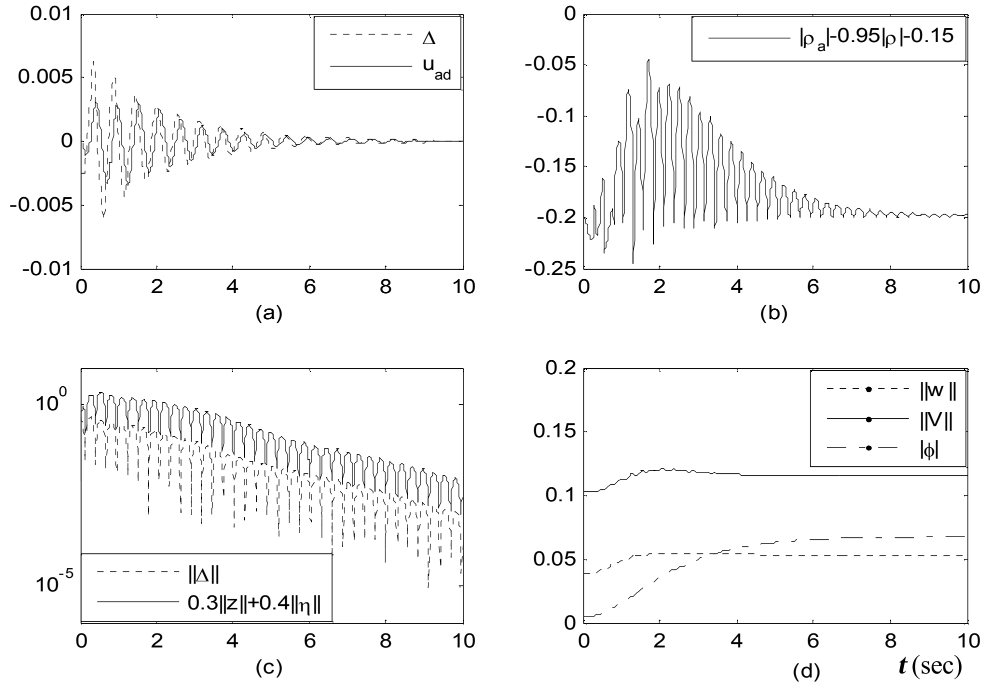

Fig 5. Stabilization of the TORA system with parameters uncertainties, (a) matched uncertainty cancellation, (b) validation of the auxiliary error bound; $|\rho_a| \le 0.15 + 0.95|\rho|$,
(c) validation of the unmatched uncertainty bound, (d) norm of the adaptive weights

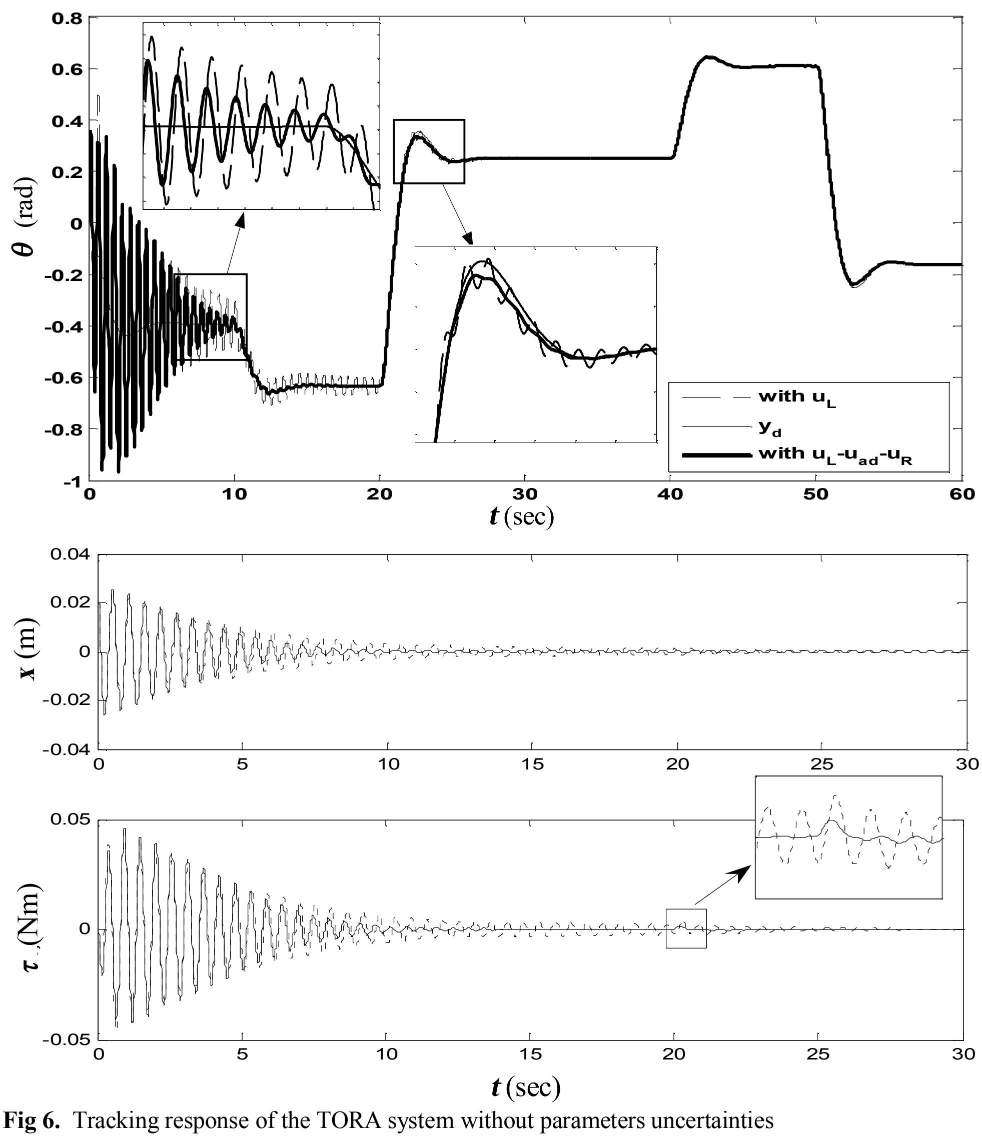

Fig 6. Tracking response of the TORA system without parameters uncertainties

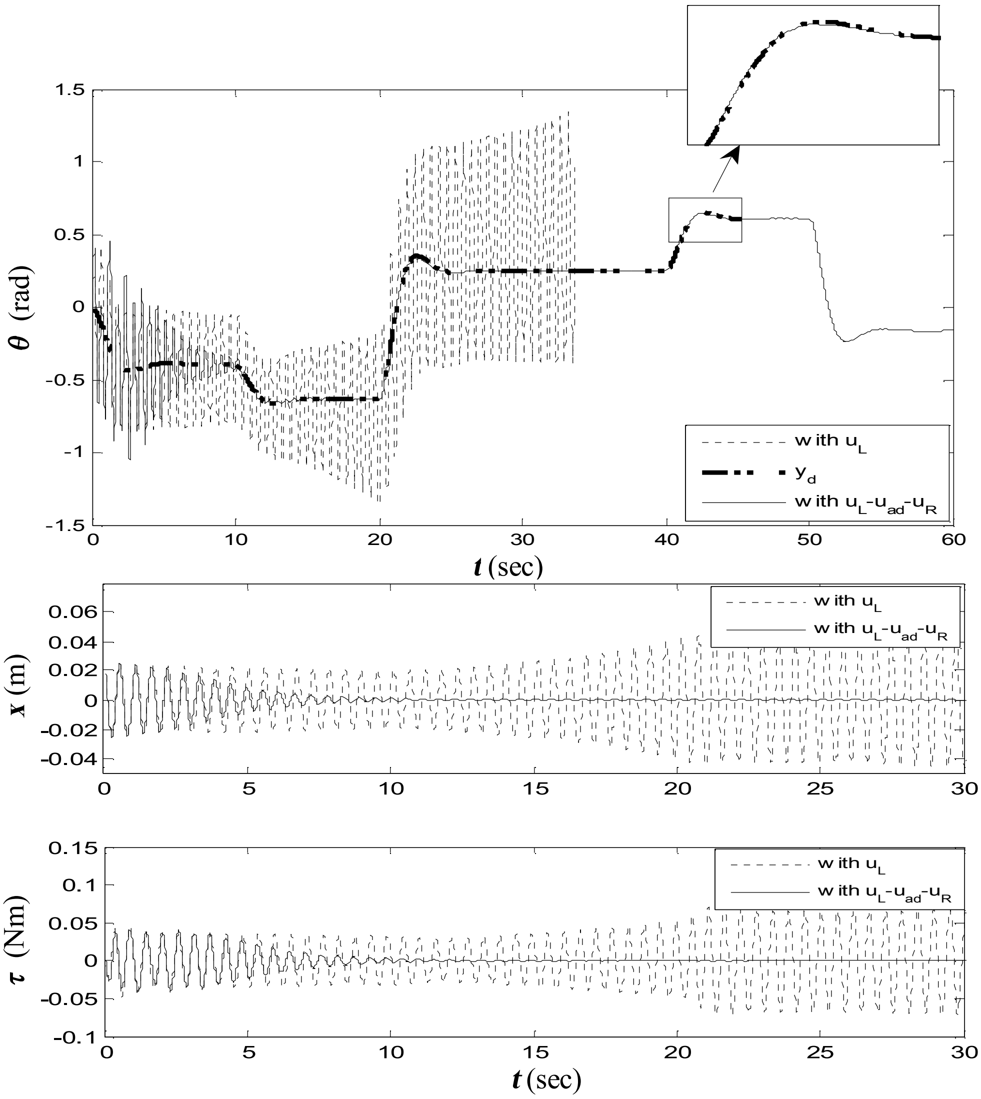

Fig 7. Tracking response of the TORA system with parameters uncertainties

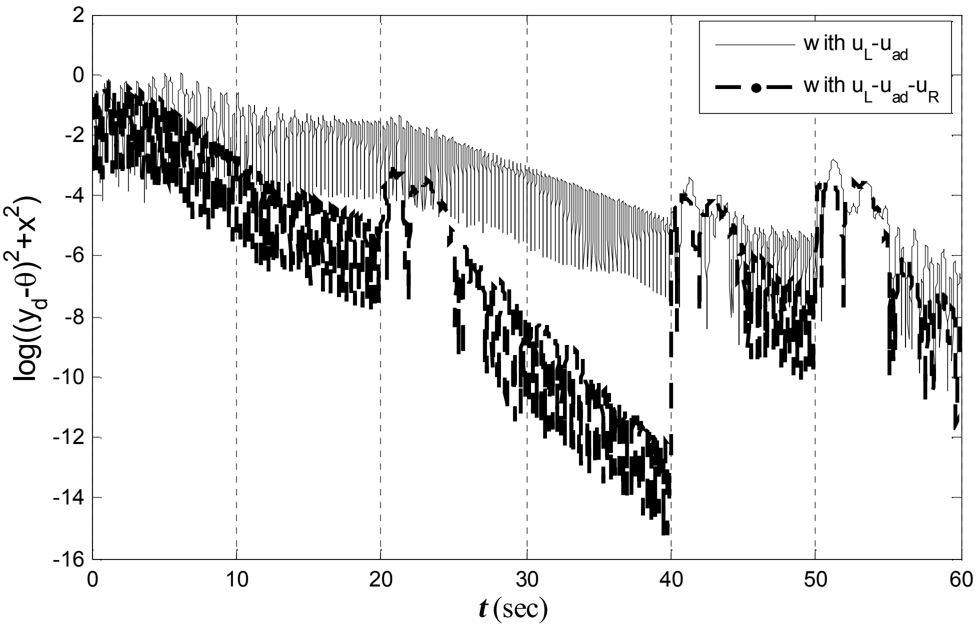

Fig. 8 Tracking response of the TORA system with parameters uncertainties, error convergence with and without u_R